# Basics of Fluid Mechanics

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Version (0.3.0.4    February 23, 2011)

‘We are like dwarfs sitting on the shoulders of giants”

from The Metalogicon by John in 1159

# CONTENTS

# LIST OF FIGURES

# LIST OF TABLES

# NOMENCLATURE

| | |
|---|---|
| $\bar{R}$ | Universal gas constant, see equation (2.26), page 49 |
| $\boldsymbol{\tau}$ | The shear stress Tenser, see equation (6.7), page 174 |
| $\ell$ | Units length., see equation (2.1), page 45 |
| $\lambda$ | bulk viscosity, see equation (8.101), page 256 |
| $\mathfrak{M}$ | Angular Momentum, see equation (6.38), page 191 |
| $\mu$ | viscosity at input temperature T, see equation (1.17), page 12 |
| $\mu_0$ | reference viscosity at reference temperature, $T_{i0}$, see equation (1.17), page 12 |
| $\boldsymbol{F}_{ext}$ | External forces by non–fluids means, see equation (6.11), page 175 |
| $\boldsymbol{U}$ | The velocity taken with the direction, see equation (6.1), page 173 |
| $\Xi$ | Martinelli parameter, see equation (9.43), page 291 |
| $A$ | The area of surface, see equation (4.136), page 108 |
| $a$ | The acceleration of object or system, see equation (4.0), page 71 |
| $B_f$ | Body force, see equation (2.9), page 47 |
| $c.v.$ | subscribe for control volume, see equation (5.0), page 146 |
| $C_p$ | Specific pressure heat, see equation (2.23), page 49 |
| $C_v$ | Specific volume heat, see equation (2.22), page 49 |
| $E_U$ | Internal energy, see equation (2.3), page 46 |

$E_u$ Internal Energy per unit mass, see equation (2.6), page 46

$E_i$ System energy at state i, see equation (2.2), page 46

$G$ The gravitation constant, see equation (4.67), page 90

$g_{\mathsf{G}}$ general Body force, see equation (4.0), page 71

$H$ Enthalpy, see equation (2.18), page 48

$h$ Specific enthalpy, see equation (2.18), page 48

$k$ the ratio of the specific heats, see equation (2.24), page 49

$k_T$ Fluid thermal conductivity, see equation (7.3), page 200

$L$ Angular momentum, see equation (3.40), page 66

$P_{atmos}$ Atmospheric Pressure, see equation (4.104), page 101

$q$ Energy per unit mass, see equation (2.6), page 46

$Q_{12}$ The energy transfered to the system between state 1 and state 2, see equation (2.2), page 46

$R$ Specific gas constant, see equation (2.27), page 50

$S$ Entropy of the system, see equation (2.13), page 48

$Suth$ Suth is Sutherland's constant and it is presented in the Table 1.1, see equation (1.17), page 12

$T_\tau$ Torque, see equation (3.42), page 66

$T_{i0}$ reference temperature in degrees Kelvin, see equation (1.17), page 12

$T_{in}$ input temperature in degrees Kelvin, see equation (1.17), page 12

$U$ velocity , see equation (2.4), page 46

$w$ Work per unit mass, see equation (2.6), page 46

$W_{12}$ The work done by the system between state 1 and state 2, see equation (2.2), page 46

$z$ the coordinate in $z$ direction, see equation (4.14), page 73

says Subscribe says, see equation (5.0), page 146

# The Book Change Log

## *Version 0.3.0.4*

### Feb 23, 2011 (3.5 M 392 pages)

- Insert discussion about Pushka equation and bulk modulus.
- Addition of several examples integral Energy chapter.
- English and addition of other minor exampls in various chapters.

## *Version 0.3.0.3*

### Dec 5, 2010 (3.3 M 378 pages)

- Add additional discussion about bulk modulus of geological system.
- Addition of several examples with respect speed of sound with variation density under bulk modulus. This addition was to go the compressible book and will migrate to there when the book will brought up to code.
- Brought the mass conservation chapter to code.
- additional examples in mass conservation chapter.

## *Version 0.3.0.2*

### Nov 19, 2010 (3.3 M 362 pages)

- Further improved the script for the chapter log file for latex (macro) process.
- Add discussion change of bulk modulus of mixture.

- Addition of several examples.
- Improve English in several chapters.

## *Version 0.3.0.1*

### Nov 12, 2010 (3.3 M 358 pages)

- Build the chapter log file for latex (macro) process Steven from www.artofproblemsolving.com.
- Add discussion change of density on buck modulus calculations as example as integral equation.
- Minimal discussion of converting integral equation to differential equations.
- Add several examples on surface tension.
- Improvement of properties chapter.
- Improve English in several chapters.

## *Version 0.3.0.0*

### Oct 24, 2010 (3.3 M 354 pages)

- Change the emphasis equations to new style in Static chapter.
- Add discussion about inclined manometer
- Improve many figures and equations in Static chapter.
- Add example of falling liquid gravity as driving force in presence of shear stress.
- Improve English in static and mostly in differential analysis chapter.

## *Version 0.2.9.1*

### Oct 11, 2010 (3.3 M 344 pages)

- Change the emphasis equations to new style in Thermo chapter.
- Correct the ideal gas relationship typo thanks to Michal Zadrozny.
- Add example, change to the new empheq format and improve cylinder figure.
- Add to the appendix the differentiation of vector operations.
- Minor correction to to the wording in page 11 viscosity density issue (thanks to Prashant Balan).
- Add example to dif chap on concentric cylinders poiseuille flow.

## *Version 0.2.9*

### Sep 20, 2010 (3.3 M 338 pages)

- Initial release of the differential equations chapter.
- Improve the emphasis macro for the important equation and useful equation.

## *Version 0.2.6*

### March 10, 2010 (2.9 M 280 pages)

- add example to Mechanical Chapter and some spelling corrected.

## *Version 0.2.4*

### March 01, 2010 (2.9 M 280 pages)

- The energy conservation chapter was released.
- Some additions to mass conservation chapter on averaged velocity.
- Some additions to momentum conservation chapter.
- Additions to the mathematical appendix on vector algebra.
- Additions to the mathematical appendix on variables separation in second order ode equations.
- Add the macro protect to insert figure in lower right corner thanks to Steven from www.artofproblemsolving.com.
- Add the macro to improve emphases equation thanks to Steven from www.artofproblemsolving.com.
- Add example about the the third component of the velocity.
- English corrections, Thanks to Eliezer Bar-Meir

## *Version 0.2.3*

### Jan 01, 2010 (2.8 M 241 pages)

- The momentum conservation chapter was released.
- Corrections to Static Chapter.
- Add the macro ekes to equations in examples thanks to Steven from www.artofproblemsolving.com.
- English corrections, Thanks to Eliezer Bar-Meir

## *Version 0.1.9*

### Dec 01, 2009 (2.6 M 219 pages)

- The mass conservation chapter was released.
- Add Reynold's Transform explanation.
- Add example on angular rotation to statics chapter.
- Add the open question concept. Two open questions were released.
- English corrections, Thanks to Eliezer Bar-Meir

## *Version 0.1.8.5*

### Nov 01, 2009 (2.5 M 203 pages)

- First true draft for the mass conservation.
- Improve the dwarfing macro to allow flexibility with sub title.
- Add the first draft of the temperature-velocity diagram to the Therm's chapter.

## *Version 0.1.8.1*

### Sep 17, 2009 (2.5 M 197 pages)

- Continue fixing the long titles issues.
- Add some examples to static chapter.
- Add an example to mechanics chapter.

## *Version 0.1.8a*

### July 5, 2009 (2.6 M 183 pages)

- Fixing some long titles issues.
- Correcting the gas properties tables (thanks to Heru and Micheal)
- Move the gas tables to common area to all the books.

## *Version 0.1.8*

### Aug 6, 2008 (2.4 M 189 pages)

- Add the chapter on introduction to muli–phase flow

- Again additional improvement to the index (thanks to Irene).
- Add the Rayleigh–Taylor instability.
- Improve the doChap scrip to break up the book to chapters.

## *Version 0.1.6*

### Jun 30, 2008 (1.3 M 151 pages)

- Fix the English in the introduction chapter, (thanks to Tousher).
- Improve the Index (thanks to Irene).
- Remove the multiphase chapter (it is not for public consumption yet).

## *Version 0.1.5a*

### Jun 11, 2008 (1.4 M 155 pages)

- Add the constant table list for the introduction chapter.
- Fix minor issues (English) in the introduction chapter.

## *Version 0.1.5*

### Jun 5, 2008 (1.4 M 149 pages)

- Add the introduction, viscosity and other properties of fluid.
- Fix very minor issues (English) in the static chapter.

## *Version 0.1.1*

### May 8, 2008 (1.1 M 111 pages)

- Major English corrections for the three chapters.
- Add the product of inertia to mechanics chapter.
- Minor corrections for all three chapters.

Version 0.1a April 23, 2008

*Version 0.1a*

**April 23, 2008**

- The Thermodynamics chapter was released.
- The mechanics chapter was released.
- The static chapter was released (the most extensive and detailed chapter).

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- **Date(s) of contribution(s):** 1999 to present
- **Nature of contribution:** Original author.
- **Contact at:** `barmeir at gmail.com`

### Steven from artofproblemsolving.com

- **Date(s) of contribution(s):** June 2005, Dec, 2009

- **Nature of contribution:** LaTeX formatting, help on building the useful equation and important equation macros.
- **Nature of contribution:** In 2009 creating the exEq macro to have different counter for example.

### Dan H. Olson

- **Date(s) of contribution(s):** April 2008
- **Nature of contribution:** Some discussions about chapter on mechanics and correction of English.

### Richard Hackbarth

- **Date(s) of contribution(s):** April 2008
- **Nature of contribution:** Some discussions about chapter on mechanics and correction of English.

### John Herbolenes

- **Date(s) of contribution(s):** August 2009
- **Nature of contribution:** Provide some example for the static chapter.

### Eliezer Bar-Meir

- **Date(s) of contribution(s):** Nov 2009, Dec 2009
- **Nature of contribution:** Correct many English mistakes Mass.
- **Nature of contribution:** Correct many English mistakes Momentum.

### Henry Schoumertate

- **Date(s) of contribution(s):** Nov 2009
- **Nature of contribution:** Discussion on the mathematics of Reynolds Transforms.

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## Typo corrections and other "minor" contributions

- **R. Gupta**, January 2008, help with the original **img** macro and other ( LaTeX issues).
- **Tousher Yang** April 2008, review of statics and thermo chapters.
- Corretion to equation (2.38) by Michal Zadrozny. (Nov 2010) Corretion to wording in viscosity density Prashant Balan. (Nov 2010)

# About This Author

Genick Bar-Meir holds a Ph.D. in Mechanical Engineering from University of Minnesota and a Master in Fluid Mechanics from Tel Aviv University. Dr. Bar-Meir was the last student of the late Dr. R.G.E. Eckert. Much of his time has been spend doing research in the field of heat and mass transfer (related to renewal energy issues) and this includes fluid mechanics related to manufacturing processes and design. Currently, he spends time writing books (there are already three very popular books) and softwares for the POTTO project (see Potto Prologue). The author enjoys to encourage his students to understand the material beyond the basic requirements of exams.

In his early part of his professional life, Bar-Meir was mainly interested in elegant models whether they have or not a practical applicability. Now, this author's views had changed and the virtue of the practical part of any model becomes the essential part of his ideas, books and software.

He developed models for Mass Transfer in high concentration that became a building blocks for many other models. These models are based on analytical solution to a family of equations[1]. As the change in the view occurred, Bar-Meir developed models that explained several manufacturing processes such the rapid evacuation of gas from containers, the critical piston velocity in a partially filled chamber (related to hydraulic jump), application of supply and demand to rapid change power system and etc. All the models have practical applicability. These models have been extended by several research groups (needless to say with large research grants). For example, the Spanish Comision Interministerial provides grants TAP97-0489 and PB98-0007, and the CICYT and the European Commission provides 1FD97-2333 grants for minor aspects of that models. Moreover, the author's models were used in numerical works, in GM, British industry, Spain, and Canada.

In the area of compressible flow, it was commonly believed and taught that there is only weak and strong shock and it is continue by Prandtl–Meyer function. Bar–

[1]Where the mathematicians were able only to prove that the solution exists.

Meir discovered the analytical solution for oblique shock and showed that there is a quiet buffer between the oblique shock and Prandtl–Meyer. He also build analytical solution to several moving shock cases. He described and categorized the filling and evacuating of chamber by compressible fluid in which he also found analytical solutions to cases where the working fluid was ideal gas. The common explanation to Prandtl–Meyer function shows that flow can turn in a sharp corner. Engineers have constructed design that based on this conclusion. Bar-Meir demonstrated that common Prandtl–Meyer explanation violates the conservation of mass and therefor the turn must be around a finite radius. The author's explanations on missing diameter and other issues in fanno flow and ““naughty professor's question”” are used in the industry.

In his book “Basics of Fluid Mechanics”, Bar-Meir demonstrated that fluids must have wavy surface when the materials flow together. All the previous models for the flooding phenomenon did not have a physical explanation to the dryness. He built a model to explain the flooding problem (two phase flow) based on the physics. He also constructed and explained many new categories for two flow regimes.

The author lives with his wife and three children. A past project of his was building a four stories house, practically from scratch. While he writes his programs and does other computer chores, he often feels clueless about computers and programing. While he is known to look like he knows about many things, the author just know to learn quickly. The author spent years working on the sea (ships) as a engine sea officer but now the author prefers to remain on solid ground.

# Prologue For The POTTO Project

This books series was born out of frustrations in two respects. The first issue is the enormous price of college textbooks. It is unacceptable that the price of the college books will be over $150 per book (over 10 hours of work for an average student in The United States).

The second issue that prompted the writing of this book is the fact that we as the public have to deal with a corrupted judicial system. As individuals we have to obey the law, particularly the copyright law with the "infinite[2]" time with the copyright holders. However, when applied to "small" individuals who are not able to hire a large legal firm, judges simply manufacture facts to make the little guy lose and pay for the defense of his work. On one hand, the corrupted court system defends the "big" guys and on the other hand, punishes the small "entrepreneur" who tries to defend his or her work. It has become very clear to the author and founder of the POTTO Project that this situation must be stopped. Hence, the creation of the POTTO Project. As R. Kook, one of this author's sages, said instead of whining about arrogance and incorrectness, one should increase wisdom. This project is to increase wisdom and humility.

The Potto Project has far greater goals than simply correcting an abusive Judicial system or simply exposing abusive judges. It is apparent that writing textbooks especially for college students as a cooperation, like an open source, is a new idea[3]. Writing a book in the technical field is not the same as writing a novel. The writing of a technical book is really a collection of information and practice. There is always someone who can add to the book. The study of technical material isn't only done by having to memorize the material, but also by coming to understand and be able to solve

[2]After the last decision of the Supreme Court in the case of Eldred v. Ashcroff (see `http://cyber.law.harvard.edu/openlaw/eldredvashcroft` for more information) copyrights practically remain indefinitely with the holder (not the creator).

[3]In some sense one can view the encyclopedia Wikipedia as an open content project (see `http://en.wikipedia.org/wiki/Main_Page`). The wikipedia is an excellent collection of articles which are written by various individuals.

related problems. The author has not found any technique that is more useful for this purpose than practicing the solving of problems and exercises. One can be successful when one solves as many problems as possible. To reach this possibility the collective book idea was created/adapted. While one can be as creative as possible, there are always others who can see new aspects of or add to the material. The collective material is much richer than any single person can create by himself.

The following example explains this point: The army ant is a kind of carnivorous ant that lives and hunts in the tropics, hunting animals that are even up to a hundred kilograms in weight. The secret of the ants' power lies in their collective intelligence. While a single ant is not intelligent enough to attack and hunt large prey, the collective power of their networking creates an extremely powerful intelligence to carry out this attack[4]. When an insect which is blind can be so powerful by networking, so can we in creating textbooks by this powerful tool.

Why would someone volunteer to be an author or organizer of such a book? This is the first question the undersigned was asked. The answer varies from individual to individual. It is hoped that because of the open nature of these books, they will become the most popular books and the most read books in their respected field. For example, the books on compressible flow and die casting became the most popular books in their respective area. In a way, the popularity of the books should be one of the incentives for potential contributors. The desire to be an author of a well–known book (at least in his/her profession) will convince some to put forth the effort. For some authors, the reason is the pure fun of writing and organizing educational material. Experience has shown that in explaining to others any given subject, one also begins to better understand the material. Thus, contributing to these books will help one to understand the material better. For others, the writing of or contributing to this kind of books will serve as a social function. The social function can have at least two components. One component is to come to know and socialize with many in the profession. For others the social part is as simple as a desire to reduce the price of college textbooks, especially for family members or relatives and those students lacking funds. For some contributors/authors, in the course of their teaching they have found that the textbook they were using contains sections that can be improved or that are not as good as their own notes. In these cases, they now have an opportunity to put their notes to use for others. Whatever the reasons, the undersigned believes that personal intentions are appropriate and are the author's/organizer's private affair.

If a contributor of a section in such a book can be easily identified, then that contributor will be the copyright holder of that specific section (even within question/answer sections). The book's contributor's names could be written by their sections. It is not just for experts to contribute, but also students who happened to be doing their homework. The student's contributions can be done by adding a question and perhaps the solution. Thus, this method is expected to accelerate the creation of these high quality books.

These books are written in a similar manner to the open source software

[4]see also in Franks, Nigel R.; "Army Ants: A Collective Intelligence," American Scientist, 77:139, 1989 (see for information `http://www.ex.ac.uk/bugclub/raiders.html`)

process. Someone has to write the skeleton and hopefully others will add "flesh and skin." In this process, chapters or sections can be added after the skeleton has been written. It is also hoped that others will contribute to the question and answer sections in the book. But more than that, other books contain data[5] which can be typeset in LaTeX. These data (tables, graphs and etc.) can be redone by anyone who has the time to do it. Thus, the contributions to books can be done by many who are not experts. Additionally, contributions can be made from any part of the world by those who wish to translate the book.

It is hoped that the books will be error-free. Nevertheless, some errors are possible and expected. Even if not complete, better discussions or better explanations are all welcome to these books. These books are intended to be "continuous" in the sense that there will be someone who will maintain and improve the books with time (the organizer(s)).

These books should be considered more as a project than to fit the traditional definition of "plain" books. Thus, the traditional role of author will be replaced by an organizer who will be the one to compile the book. The organizer of the book in some instances will be the main author of the work, while in other cases only the gate keeper. This may merely be the person who decides what will go into the book and what will not (gate keeper). Unlike a regular book, these works will have a version number because they are alive and continuously evolving.

In the last 5 years three textbooks have been constructed which are available for download. These books contain innovative ideas which make some chapters the best in the world. For example, the chapters on Fanno flow and Oblique shock contain many original ideas such as the full analytical solution to the oblique shock, many algorithms for calculating Fanno flow parameters which are not found in any other book. In addition, Potto has auxiliary materials such as the gas dynamics tables (the largest compressible flow tables collection in the world), Gas Dynamics Calculator (Potto-GDC), etc.

The combined number downloads of these books is over half a million (December 2009) or in a rate of 20,000 copies a month. Potto books on compressible flow and fluid mechanics are used as the main textbook or as a reference book in several universities around the world. The books are used in more than 165 different countries around the world. Every month people from about 110 different countries download these books. The book on compressible flow is also used by "young engineers and scientists" in NASA according to Dr. Farassat, NASA Langley Research Center.

The undersigned of this document intends to be the organizer/author/coordinator of the projects in the following areas:

[5] Data are not copyrighted.

*Table -1. Books under development in Potto project.*

| Project Name | Progress | Remarks | Version | Availability for Public Download |
|---|---|---|---|---|
| Compressible Flow | beta | | 0.4.8.2 | ✔ |
| Die Casting | alpha | | 0.0.3 | ✔ |
| Dynamics | NSY | | 0.0.0 | ✖ |
| Fluid Mechanics | alpha | | 0.1.1 | ✔ |
| Heat Transfer | NSY | Based on Eckert | 0.0.0 | ✖ |
| Mechanics | NSY | | 0.0.0 | ✖ |
| Open Channel Flow | NSY | | 0.0.0 | ✖ |
| Statics | early alpha | first chapter | 0.0.1 | ✖ |
| Strength of Material | NSY | | 0.0.0 | ✖ |
| Thermodynamics | early alpha | | 0.0.01 | ✖ |
| Two/Multi phases flow | NSY | Tel-Aviv'notes | 0.0.0 | ✖ |

NSY = Not Started Yet

The meaning of the progress is as:

- The Alpha Stage is when some of the chapters are already in a rough draft;
- in Beta Stage is when all or almost all of the chapters have been written and are at least in a draft stage;
- in Gamma Stage is when all the chapters are written and some of the chapters are in a mature form; and
- the Advanced Stage is when all of the basic material is written and all that is left are aspects that are active, advanced topics, and special cases.

The mature stage of a chapter is when all or nearly all the sections are in a mature stage and have a mature bibliography as well as numerous examples for every section. The mature stage of a section is when all of the topics in the section are written, and all of the examples and data (tables, figures, etc.) are already presented. While some terms are defined in a relatively clear fashion, other definitions give merely a hint on the status. But such a thing is hard to define and should be enough for this stage.

The idea that a book can be created as a project has mushroomed from the open source software concept, but it has roots in the way science progresses. However, traditionally books have been improved by the same author(s), a process in which books have a new version every a few years. There are book(s) that have continued after their author passed away, i.e., the *Boundary Layer Theory* originated[6] by Hermann Schlichting but continues to this day. However, projects such as the Linux Documentation project demonstrated that books can be written as the cooperative effort of many individuals, many of whom volunteered to help.

Writing a textbook is comprised of many aspects, which include the actual writing of the text, writing examples, creating diagrams and figures, and writing the LaTeX macros[7] which will put the text into an attractive format. These chores can be done independently from each other and by more than one individual. Again, because of the open nature of this project, pieces of material and data can be used by different books.

[6] Originally authored by Dr. Schlichting, who passed way some years ago. A new version is created every several years.

[7] One can only expect that open source and readable format will be used for this project. But more than that, only LaTeX, and perhaps troff, have the ability to produce the quality that one expects for these writings. The text processes, especially LaTeX, are the only ones which have a cross platform ability to produce macros and a uniform feel and quality. Word processors, such as OpenOffice, Abiword, and Microsoft Word software, are not appropriate for these projects. Further, any text that is produced by Microsoft and kept in "Microsoft" format are against the spirit of this project In that they force spending money on Microsoft software.

# Prologue For This Book

## *Version 0.1.8 August 6, 2008*

### pages 189 size 2.6M

When this author was an undergraduate student, he spend time to study the wave phenomenon at the interface of open channel flow. This issue is related to renewal energy of extracting energy from brine solution (think about the Dead Sea, so much energy). The common explanation to the wave existence was that there is always a disturbance which causes instability. This author was bothered by this explanation. Now, in this version, it was proven that this wavy interface is created due to the need to satisfy the continuous velocity and shear stress at the interface and not a disturbance.

Potto project books are characterized by high quality which marked by presentation of the new developments and clear explanations. This explanation (on the wavy interface) demonstrates this characteristic of Potto project books. The introduction to multi–phase is another example to this quality. While it is a hard work to discover and develop and bring this information to the students, it is very satisfying for the author. The number of downloads of this book results from this quality. Even in this early development stage, number of downloads per month is about 5000 copies.

## *Version 0.1 April 22, 2008*

### pages 151 size 1.3M

The topic of fluid mechanics is common to several disciplines: mechanical engineering, aerospace engineering, chemical engineering, and civil engineering. In fact, it is also related to disciplines like industrial engineering, and electrical engineering. While the emphasis is somewhat different in this book, the common material is presented and hopefully can be used by all. One can only admire the wonderful advances done by the previous geniuses who work in this field. In this book it is hoped to insert, what and when a certain model is suitable than other models.

One of the difference in this book is the insertion of the introduction to multiphase flow. Clearly, multiphase is an advance topic. However, some minimal familiarity can be helpful for many engineers who have to deal with non pure single phase fluid.

This book is the third book in the series of POTTO project books. POTTO project books are open content textbooks so everyone are welcome to joint in. The topic of fluid mechanics was chosen just to fill the introduction chapter to compressible flow. During the writing it became apparent that it should be a book in its own right. In writing the chapter on fluid statics, there was a realization that it is the best chapter written on this topic. It is hoped that the other chapters will be as good this one.

This book is written in the spirit of my adviser and mentor E.R.G. Eckert. Eckert, aside from his research activity, wrote the book that brought a revolution in the education of the heat transfer. Up to Egret's book, the study of heat transfer was without any dimensional analysis. He wrote his book because he realized that the dimensional analysis utilized by him and his adviser (for the post doc), Ernst Schmidt, and their colleagues, must be taught in engineering classes. His book met strong criticism in which some called to "burn" his book. Today, however, there is no known place in world that does not teach according to Eckert's doctrine. It is assumed that the same kind of individual(s) who criticized Eckert's work will criticize this work. Indeed, the previous book, on compressible flow, met its opposition. For example, anonymous Wikipedia user name EMBaero claimed that the material in the book is plagiarizing, he just doesn't know from where and what. Maybe that was the reason that he felt that is okay to plagiarize the book on Wikipedia. These criticisms will not change the future or the success of the ideas in this work. As a wise person says "don't tell me that it is wrong, show me what is wrong"; this is the only reply. With all the above, it must be emphasized that this book is not expected to revolutionize the field but change some of the way things are taught.

The book is organized into several chapters which, as a traditional textbook, deals with a basic introduction to the fluid properties and concepts (under construction). The second chapter deals with Thermodynamics. The third book chapter is a review of mechanics. The next topic is statics. When the Static Chapter was written, this author did not realize that so many new ideas will be inserted into this topic. As traditional texts in this field, ideal flow will be presented with the issues of added mass and added forces (under construction). The classic issue of turbulence (and stability) will be presented. An introduction to multi–phase flow, not a traditional topic, will be presented next (again under construction). The next two chapters will deals with open channel flow and gas dynamics. At this stage, dimensional analysis will be present (again under construction).

# How This Book Was Written

This book started because I needed an introduction to the compressible flow book. After a while it seems that is easier to write a whole book than the two original planned chapters. In writing this book, it was assumed that introductory book on fluid mechanics should not contained many new ideas but should be modern in the material presentation. There are numerous books on fluid mechanics but none of which is open content. The approach adapted in this book is practical, and more hands–on approach. This statement really meant that the book is intent to be used by students to solve their exams and also used by practitioners when they search for solutions for practical problems. So, issue of proofs so and so are here only either to explain a point or have a solution of exams. Otherwise, this book avoids this kind of issues.

The structure of Hansen, Streeter and Wylie, and Shames books were adapted and used as a scaffolding for this book. This author was influenced by Streeter and Wylie book which was his undergrad textbooks. The chapters are not written in order. The first 4 chapters were written first because they were supposed to be modified and used as fluid mechanics introduction in "Fundamentals of Compressible Flow." Later, multi–phase flow chapter was written.

The presentation of some of the chapters is slightly different from other books because the usability of the computers. The book does not provide the old style graphical solution methods yet provides the graphical explanation of things.

Of course, this book was written on Linux (Micro$oftLess book). This book was written using the vim editor for editing (sorry never was able to be comfortable with emacs). The graphics were done by TGIF, the best graphic program that this author experienced so far. The figures were done by gle. The spell checking was done by ispell, and hope to find a way to use gaspell, a program that currently cannot be used on new Linux systems. The figure in cover page was created by Genick Bar-Meir, and is copyleft by him.

# Preface

*"In the beginning, the POTTO project was without form, and void; and emptiness was upon the face of the bits and files. And the Fingers of the Author moved upon the face of the keyboard. And the Author said, Let there be words, and there were words."*[8].

This book, Basics of Fluid Mechanics, describes the fundamentals of fluid mechanics phenomena for engineers and others. This book is designed to replace all introductory textbook(s) or instructor's notes for the fluid mechanics in undergraduate classes for engineering/science students but also for technical peoples. It is hoped that the book could be used as a reference book for people who have at least some basics knowledge of science areas such as calculus, physics, etc.

The structure of this book is such that many of the chapters could be usable independently. For example, if you need information about, say, statics' equations, you can read just chapter (4). I hope this makes the book easier to use as a reference manual. However, this manuscript is first and foremost a textbook, and secondly a reference manual only as a lucky coincidence.

I have tried to describe why the theories are the way they are, rather than just listing "seven easy steps" for each task. This means that a lot of information is presented which is not necessary for everyone. These explanations have been marked as such and can be skipped.[9] Reading everything will, naturally, increase your understanding of the many aspects of fluid mechanics.

This book is written and maintained on a volunteer basis. Like all volunteer work, there is a limit on how much effort I was able to put into the book and its organization. Moreover, due to the fact that English is my third language and time limitations, the explanations are not as good as if I had a few years to perfect them. Nevertheless, I believe professionals working in many engineering fields will benefit from this information. This book contains many worked examples, which can be very useful for many.

I have left some issues which have unsatisfactory explanations in the book, marked with a Mata mark. I hope to improve or to add to these areas in the near future.

[8] To the power and glory of the mighty God. This book is only to explain his power.

[9] At the present, the book is not well organized. You have to remember that this book is a work in progress.

Furthermore, I hope that many others will participate of this project and will contribute to this book (even small contributions such as providing examples or editing mistakes are needed).

I have tried to make this text of the highest quality possible and am interested in your comments and ideas on how to make it better. Incorrect language, errors, ideas for new areas to cover, rewritten sections, more fundamental material, more mathematics (or less mathematics); I am interested in it all. I am particularly interested in the best arrangement of the book. If you want to be involved in the editing, graphic design, or proofreading, please drop me a line. You may contact me via Email at "barmeir@gmail.com".

Naturally, this book contains material that never was published before (sorry cannot avoid it). This material never went through a close content review. While close content peer review and publication in a professional publication is excellent idea in theory. In practice, this process leaves a large room to blockage of novel ideas and plagiarism. If you would like be "peer reviews" or critic to my new ideas please send me your comment(s). Even reaction/comments from individuals like David Marshall[10].

Several people have helped me with this book, directly or indirectly. I would like to especially thank to my adviser, Dr. E. R. G. Eckert, whose work was the inspiration for this book. I also would like to thank to Jannie McRotien (Open Channel Flow chapter) and Tousher Yang for their advices, ideas, and assistance.

The symbol META was added to provide typographical conventions to blurb as needed. This is mostly for the author's purposes and also for your amusement. There are also notes in the margin, but those are solely for the author's purposes, ignore them please. They will be removed gradually as the version number advances.

I encourage anyone with a penchant for writing, editing, graphic ability, LaTeX knowledge, and material knowledge and a desire to provide open content textbooks and to improve them to join me in this project. If you have Internet e-mail access, you can contact me at "barmeir@gmail.com".

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[10]Dr. Marshall wrote to this author that the author should review other people work before he write any thing new (well, literature review is always good, isn't it?). Over ten individuals wrote me about this letter. I am asking from everyone to assume that his reaction was innocent one. While his comment looks like unpleasant reaction, it brought or cause the expansion of the explanation for the oblique shock. However, other email that imply that someone will take care of this author aren't appreciated.

# To Do List and Road Map

This book isn't complete and probably never will be completed. There will always new problems to add or to polish the explanations or include more new materials. Also issues that associated with the book like the software has to be improved. It is hoped the changes in TEX and LATEX related to this book in future will be minimal and minor. It is hoped that the style file will be converged to the final form rapidly. Nevertheless, there are specific issues which are on the "table" and they are described herein.

At this stage, many chapters are missing. Specific missing parts from every chapters are discussed below. These omissions, mistakes, approach problems are sometime appears in the book under the Meta simple like this

# Meta

sample this part.

# Meta End

You are always welcome to add a new material: problem, question, illustration or photo of experiment. Material can be further illuminate. Additional material can be provided to give a different angle on the issue at hand.

### Properties

The chapter isn't in development stage yet.

### Open Channel Flow

The chapter isn't in the development stage yet. Some parts were taken from Fundamentals of Die Casting Design book and are in a process of improvement.

# CHAPTER 1

# Introduction to Fluid Mechanics

## 1.1 What is Fluid Mechanics?

Fluid mechanics deals with the study of all fluids under static and dynamic situations. Fluid mechanics is a branch of continuous mechanics which deals with a relationship between forces, motions, and statical conditions in continuous material. This study area deals with many and diversified problems such as surface tension, fluid statics, flow in enclose bodies, or flow round bodies (solid or otherwise), flow stability, etc. In fact, almost any action a person is doing involves some kind of a fluid mechanics problem. Furthermore, the boundary between the solid mechanics and fluid mechanics is some kind of gray shed and not a sharp distinction (see Figure 1.1 for the complex relationships between the different branches which only part of it should be drawn in the same time.). For example, glass appears as a solid material, but a closer look reveals that the glass is a liquid with a large viscosity. A proof of the glass "liquidity" is the change of the glass thickness in high windows in European Churches after hundred years. The bottom part of the glass is thicker than the top part. Materials like sand (some call it quick sand) and grains should be treated as liquids. It is known that these materials have the ability to drown people. Even material such as aluminum just below the mushy zone also behaves as a liquid similarly to butter. After it was established that the boundaries of fluid mechanics aren't sharp, the discussion in this book is limited to simple and (mostly) Newtonian (sometimes power fluids) fluids which will be defined later.

The fluid mechanics study involve many fields that have no clear boundary between them. Researchers distinguish between orderly flow and chaotic flow as the laminar flow and the turbulent flow. The fluid mechanics can also be distinguish between a single phase flow and multiphase flow (flow made more than one phase or single distinguishable material). The last boundary (as all the boundaries in fluid mechanics)

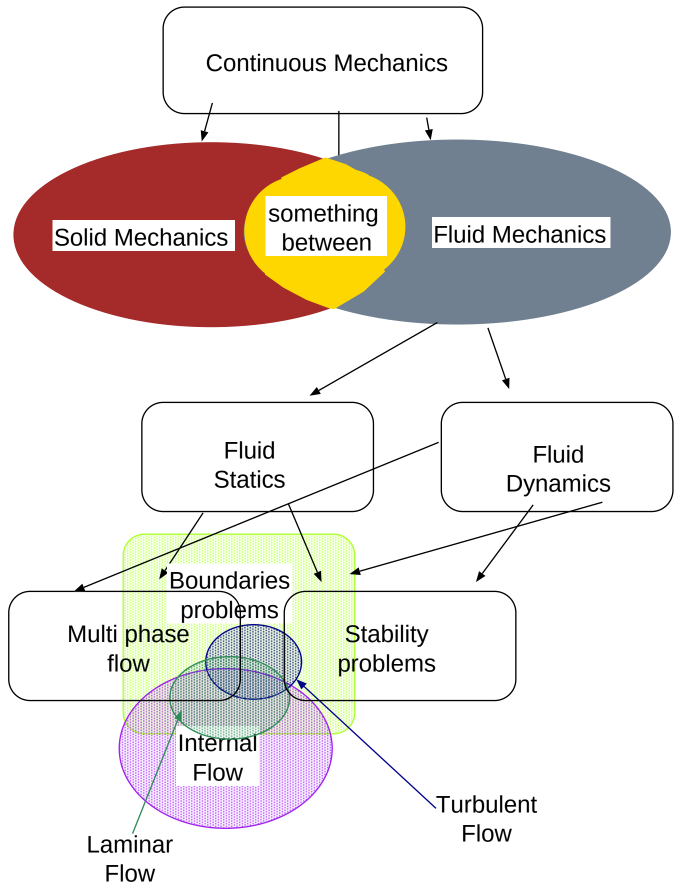



*Fig. -1.1. Diagram to explain part of relationships of fluid mechanics branches.*

isn't sharp because fluid can go through a phase change (condensation or evaporation) in the middle or during the flow and switch from a single phase flow to a multi phase flow. Moreover, flow with two phases (or materials) can be treated as a single phase (for example, air with dust particle).

After it was made clear that the boundaries of fluid mechanics aren't sharp, the study must make arbitrary boundaries between fields. Then the dimensional analysis will be used explain why in certain cases one distinguish area/principle is more relevant than the other and some effects can be neglected. Or, when a general model is need because more parameters are effecting the situation. It is this author's personal experience that the knowledge and ability to know in what area the situation lay is one of the main problems. For example, engineers in software company (EKK Inc, `http://ekkinc.com/HTML` ) analyzed a flow of a complete still liquid assuming a complex turbulent flow model. Such absurd analysis are common among engineers who do not know which model can be applied. Thus, one of the main goals of this book is to explain what model should be applied. Before dealing with the boundaries, the simplified private cases must be explained.

There are two main approaches of presenting an introduction of fluid mechanics materials. The first approach introduces the fluid kinematic and then the basic governing equations, to be followed by stability, turbulence, boundary layer and internal and external flow . The second approach deals with the Integral Analysis to be followed with Differential Analysis, and continue with Empirical Analysis. These two approaches pose a dilemma to anyone who writes an introductory book for the fluid mechanics. These two approaches have justifications and positive points. Reviewing many books on fluid mechanics made it clear, there isn't a clear winner. This book attempts to find a hybrid approach in which the kinematic is presented first (aside to standard initial four chapters) follow by Integral analysis and continued by Differential analysis. The ideal flow (frictionless flow) should be expanded compared to the regular treatment. This book is unique in providing chapter on multiphase flow. Naturally, chapters on open channel flow (as a sub class of the multiphase flow) and compressible flow (with the latest developments) are provided.

## 1.2 Brief History

The need to have some understanding of fluid mechanics started with the need to obtain water supply. For example, people realized that wells have to be dug and crude pumping devices need to be constructed. Later, a large population created a need to solve waste (sewage) and some basic understanding was created. At some point, people realized that water can be used to move things and provide power. When cities increased to a larger size, aqueducts were constructed. These aqueducts reached their greatest size and grandeur in those of the City of Rome and China.

Yet, almost all knowledge of the ancients can be summarized as application of instincts, with the exception Archimedes (250 B.C.) on the principles of buoyancy. For example, larger tunnels built for a larger water supply, etc. There were no calculations even with the great need for water supply and transportation. The first progress in fluid mechanics was made by Leonardo Da Vinci (1452-1519) who built the first chambered canal lock near Milan. He also made several attempts to study the flight (birds) and developed some concepts on the origin of the forces. After his initial work, the knowledge of fluid mechanics (hydraulic) increasingly gained speed by the contributions of Galileo, Torricelli, Euler, Newton, Bernoulli family, and D'Alembert. At that stage theory and experiments had some discrepancy. This fact was acknowledged by D'Alembert who stated that, "The theory of fluids must necessarily be based upon experiment." For example the concept of ideal liquid that leads to motion with no resistance, conflicts with the reality.

This discrepancy between theory and practice is called the "D'Alembert paradox" and serves to demonstrate the limitations of theory alone in solving fluid problems. As in thermodynamics, two different of school of thoughts were created: the first believed that the solution will come from theoretical aspect alone, and the second believed that solution is the pure practical (experimental) aspect of fluid mechanics. On the theoretical side, considerable contribution were made by Euler, La Grange, Helmhoitz, Kirchhoff, Rayleigh, Rankine, and Kelvin. On the "experimental" side, mainly in pipes and open channels area, were Brahms, Bossut, Chezy, Dubuat, Fabre, Coulomb, Dupuit, d'Aubisson, Hagen, and Poisseuille.

In the middle of the nineteen century, first Navier in the molecular level and later Stokes from continuous point of view succeeded in creating governing equations for real fluid motion. Thus, creating a matching between the two school of thoughts: experimental and theoretical. But, as in thermodynamics, people cannot relinquish control. As results it created today "strange" names: Hydrodynamics, Hydraulics, Gas Dynamics, and Aeronautics.

The Navier-Stokes equations, which describes the flow (or even Euler equations), were considered unsolvable during the mid nineteen century because of the high complexity. This problem led to two consequences. Theoreticians tried to simplify the equations and arrive at approximated solutions representing specific cases. Examples of such work are Hermann von Helmholtz's concept of vortexes (1858), Lanchester's concept of circulatory flow (1894), and the Kutta-Joukowski circulation theory of lift (1906). The experimentalists, at the same time proposed many correlations to many fluid mechanics problems, for example, resistance by Darcy, Weisbach, Fanning, Ganguillet, and Manning. The obvious happened without theoretical guidance, the empirical formulas generated by fitting curves to experimental data (even sometime merely presenting the results in tabular form) resulting in formulas that the relationship between the physics and properties made very little sense.

At the end of the twenty century, the demand for vigorous scientific knowledge that can be applied to various liquids as opposed to formula for every fluid was created by the expansion of many industries. This demand coupled with new several novel concepts like the theoretical and experimental researches of Reynolds, the development of dimensional analysis by Rayleigh, and Froude's idea of the use of models change the science of the fluid mechanics. Perhaps the most radical concept that effects the fluid mechanics is of Prandtl's idea of boundary layer which is a combination of the modeling and dimensional analysis that leads to modern fluid mechanics. Therefore, many call Prandtl as the father of modern fluid mechanics. This concept leads to mathematical basis for many approximations. Thus, Prandtl and his students Blasius, von Karman, Meyer, and Blasius and several other individuals as Nikuradse, Rose, Taylor, Bhuckingham, Stanton, and many others, transformed the fluid mechanics to modern science that we have known today.

While the understanding of the fundamentals did not change much, after World War Two, the way how it was calculated changed. The introduction of the computers during the 60s and much more powerful personal computer has changed the field. There are many open source programs that can analyze many fluid mechanics situations. Today many problems can be analyzed by using the numerical tools and provide reasonable results. These programs in many cases can capture all the appropriate parameters and adequately provide a reasonable description of the physics. However, there are many other cases that numerical analysis cannot provide any meaningful result (trends). For example, no weather prediction program can produce good engineering quality results (where the snow will fall within 50 kilometers accuracy. Building a car with this accuracy is a disaster). In the best scenario, these programs are as good as the input provided. Thus, assuming turbulent flow for still flow simply provides erroneous results (see for example, EKK, Inc).

## 1.3 Kinds of Fluids

Some differentiate fluid from solid by the reaction to shear stress. It is a known fact said that the fluid continuously and permanently deformed under shear stress while solid exhibits a finite deformation which does not change with time. It is also said that liquid cannot return to their original state after the deformation. This differentiation leads to three groups of materials: solids and liquids. This test creates a new material group that shows dual behaviors; under certain limits; it behaves like solid and under others it behaves like liquid (see Figure 1.1). The study of this kind of material called rheology and it will (almost) not be discussed in this book. It is evident from this discussion that when a liquid is at rest, no shear stress is applied.

The fluid is mainly divided into two categories: liquids and gases. The main difference between the liquids and gases state is that gas will occupy the whole volume while liquids has an almost fix volume. This difference can be, for most practical purposes considered, sharp even though in reality this difference isn't sharp. The difference between a gas phase to a liquid phase above the critical point are practically minor. But below the critical point, the change of water pressure by 1000% only change the volume by less than 1 percent. For example, a change in the volume by more 5% will required tens of thousands percent change of the pressure. So, if the change of pressure is significantly less than that, then the change of volume is at best 5%. Hence, the pressure will not affect the volume. In gaseous phase, any change in pressure directly affects the volume. The gas fills the volume and liquid cannot. Gas has no free interface/surface (since it does fill the entire volume).

There are several quantities that have to be addressed in this discussion. The first is **force** which was reviewed in physics. The unit used to measure is [N]. It must be remember that force is a vector, e.g it has a direction. The second quantity discussed here is the area. This quantity was discussed in physics class but here it has an additional meaning, and it is referred to the direction of the area. The direction of area is perpendicular to the area. The area is measured in $[m^2]$. Area of three–dimensional object has no single direction. Thus, these kinds of areas should be addressed infinitesimally and locally.

The traditional quantity, which is force per area has a new meaning. This is a result of division of a vector by a vector and it is referred to as tensor. In this book, the emphasis is on the physics, so at this stage the tensor will have to be broken into its components. Later, the discussion on the mathematical meaning will be presented (later version). For the discussion here, the pressure has three components, one in the area direction and two perpendicular to the area. The pressure component in the area direction is called pressure (great way to confuse, isn't it?). The other two components are referred as the shear stresses. The units used for the pressure components is $[N/m^2]$.

The density is a property which requires that liquid to be continuous. The density can be changed and it is a function of time and space (location) but must have a continues property. It doesn't mean that a sharp and abrupt change in the density cannot occur. It referred to density that is independent of the sampling size. Figure 1.2 shows the density as a function of the sample size. After certain sample size, the density remains constant. Thus, the density is defined as

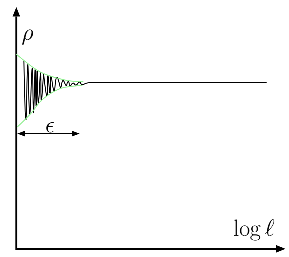


*Fig. -1.2. Density as a function of the size of sample.*

$$\rho = \lim_{\Delta V \longrightarrow \varepsilon} \frac{\Delta m}{\Delta V} \tag{1.1}$$

It must be noted that $\varepsilon$ is chosen so that the continuous assumption is not broken, that is, it did not reach/reduced to the size where the atoms or molecular statistical calculations are significant (see Figure 1.2 for point where the green lines converge to constant density). When this assumption is broken, then, the principles of statistical mechanics must be utilized.

## 1.4 Shear Stress

The shear stress is part of the pressure tensor. However, here it will be treated as a separate issue. In solid mechanics, the shear stress is considered as the ratio of the force acting on area in the direction of the forces perpendicular to area. Different from solid, fluid cannot pull directly but through a solid surface. Consider liquid that undergoes a shear stress between a short distance of two plates as shown in Figure (1.3).

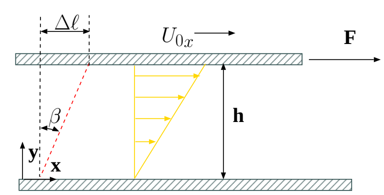


*Fig. -1.3. Schematics to describe the shear stress in fluid mechanics.*

The upper plate velocity generally will be

$$U = f(A, F, h) \tag{1.2}$$

Where $A$ is the area, the $F$ denotes the force, $h$ is the distance between the plates. From solid mechanics study, it was shown that when the force per area increases, the velocity of the plate increases also. Experiments show that the increase of height will increase the velocity up to a certain range. Consider moving the plate with a zero lubricant ($h \sim 0$) (results in large force) or a large amount of lubricant (smaller force). In this discussion, the aim is to develop differential equation, thus the small distance analysis is applicable.

For cases where the dependency is linear, the following can be written

$$U \propto \frac{h\,F}{A} \tag{1.3}$$

Equations (1.3) can be rearranged to be

$$\frac{U}{h} \propto \frac{F}{A} \tag{1.4}$$

Shear stress was defined as

$$\tau_{xy} = \frac{F}{A} \tag{1.5}$$

From equations (1.4) and (1.5) it follows that ratio of the velocity to height is proportional to shear stress. Hence, applying the coefficient to obtain a new equality as

$$\tau_{xy} = \mu\,\frac{U}{h} \tag{1.6}$$

Where $\mu$ is called the absolute viscosity or dynamic viscosity which will be discussed later in this chapter in great length.

In steady state, the distance the upper plate moves after small amount of time, $\delta t$ is

$$d\ell = U\,\delta t \tag{1.7}$$

From Figure 1.4 it can be noticed that for a small angle, the regular approximation provides

$$d\ell = U\,\delta t = \overbrace{h\,\delta\beta}^{geometry} \tag{1.8}$$

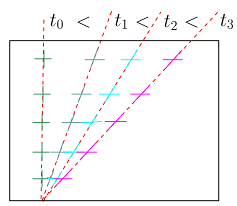



*Fig. -1.4. The deformation of fluid due to shear stress as progression of time.*

From equation (1.8) it follows that

$$U = h\frac{\delta\beta}{\delta t} \tag{1.9}$$

Combining equation (1.9) with equation (1.6) yields

$$\tau_{xy} = \mu\,\frac{\delta\beta}{\delta t} \tag{1.10}$$

If the velocity profile is linear between the plate (it will be shown later that it is consistent with derivations of velocity), then it can be written for small a angel that

$$\frac{\delta\beta}{\delta t} = \frac{dU}{dy} \tag{1.11}$$

Materials which obey equation (1.10) referred to as Newtonian fluid. For this kind of substance

$$\tau_{xy} = \mu \frac{dU}{dy} \tag{1.12}$$

Newtonian fluids are fluids which the ratio is constant. Many fluids fall into this category such as air, water etc. This approximation is appropriate for many other fluids but only within some ranges.

Equation (1.9) can be interpreted as momentum in the $x$ direction transferred into the $y$ direction. Thus, the viscosity is the resistance to the flow (flux) or the movement. The property of viscosity, which is exhibited by all fluids, is due to the existence of cohesion and interaction between fluid molecules. These cohesion and interactions hamper the flux in y–direction. Some referred to shear stress as viscous flux of x–momentum in the y–direction. The units of shear stress are the same as flux per time as following

$$\frac{F}{A}\left[\frac{kg\, m}{sec^2}\,\frac{1}{m^2}\right] = \frac{\dot{m}\, U}{A}\left[\frac{kg}{sec}\,\frac{m}{sec}\,\frac{1}{m^2}\right]$$

Thus, the notation of $\tau_{xy}$ is easier to understand and visualize. In fact, this interpretation is more suitable to explain the molecular mechanism of the viscosity. The units of absolute viscosity are $[N\, sec/m^2]$.

Example 1.1:

*A space of 1 [cm] width between two large plane surfaces is filled with glycerin. Calculate the force that is required to drag a very thin plate of 1 [$m^2$] at a speed of 0.5 m/sec. It can be assumed that the plates remains in equidistant from each other and steady state is achieved instantly.*

<u>Solution</u>

Assuming Newtonian flow, the following can be written (see equation (1.6))

$$F = \frac{A\, \mu U}{h} \sim \frac{1 \times 1.069 \times 0.5}{0.01} = 53.45[N]$$

End Solution

Example 1.2:

*Castor oil at $25°C$ fills the space between two concentric cylinders of 0.2[m] and 0.1[m] diameters with height of 0.1 [m]. Calculate the torque required to rotate the inner cylinder at 12 rpm, when the outer cylinder remains stationary. Assume steady state conditions.*

<u>Solution</u>

The velocity is

$$U = r\,\dot{\theta} = 2\,\pi\,r_i\,\text{rps} = 2\times\pi\times 0.1\times\overbrace{12/60}^{\text{rps}} = 0.4\,\pi\,r_i$$

Where $rps$ is revolution per second.

The same way as in example (1.1), the moment can be calculated as the force times the distance as

$$M = F\,\ell = \frac{\overbrace{\ell}^{r_i}\;\overbrace{A}^{2\,\pi\,r_i\,h}\;\mu U}{r_o - r_i}$$

In this case $r_o - r_i = h$ thus,

$$M = \frac{2\,\pi^2\;\overbrace{0.1^3}^{r_i}\;\cancel{h}\;\overbrace{0.986}^{\mu}\;0.4}{\cancel{h}} \sim .0078[N\,m]$$

End Solution

## 1.5 Viscosity

### 1.5.1 General

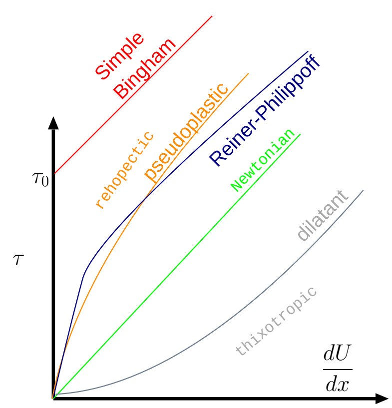



*Fig. -1.5. The different of power fluids families.*

Viscosity varies widely with temperature. However, temperature variation has an opposite effect on the viscosities of liquids and gases. The difference is due to their fundamentally different mechanism creating viscosity characteristics. In gases, molecules are sparse and cohesion is negligible, while in the liquids, the molecules are more compact and cohesion is more dominate. Thus, in gases, the exchange of momentum between layers brought as a result of molecular movement normal to the general direction of flow, and it resists the flow. This molecular activity is known to increase with temperature, thus, the viscosity of gases will increase with temperature. This reasoning is a result of the considerations of the kinetic theory. This theory indicates that gas viscosities vary directly with the square root of temperature. In liquids, the momentum exchange due to molecular movement is small compared to the cohesive forces between the molecules. Thus, the viscosity is primarily dependent on the magnitude of these cohesive forces. Since these forces decrease rapidly with increases of temperature, liquid viscosities decrease as temperature increases.

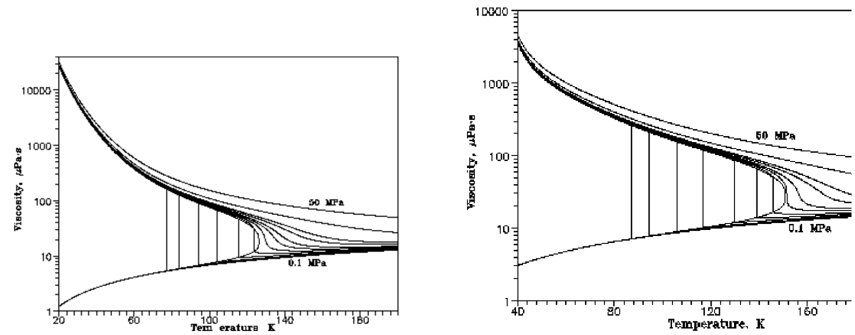



*Fig. -1.6. Nitrogen (left) and Argon (right) viscosity as a function of the temperature and pressure after Lemmon and Jacobsen.*

Figure 1.6 demonstrates that viscosity increases slightly with pressure, but this variation is negligible for most engineering problems. Well above the critical point, both materials are only a function of the temperature. On the liquid side below the critical point, the pressure has minor effect on the viscosity. It must be stress that the viscosity in the dome is meaningless. There is no such a thing of viscosity at 30% liquid. It simply depends on the structure of the flow as will be discussed in the chapter on multi phase flow. The lines in the above diagrams are only to show constant pressure lines. Oils have the greatest increase of viscosity with pressure which is a good thing for many engineering purposes.

### 1.5.2 Non–Newtonian Fluids

In equation (1.5), the relationship between the velocity and the shear stress was assumed to be linear. Not all the materials obey this relationship. There is a large class of materials which shows a non-linear relationship with velocity for any shear stress. This class of materials can be approximated by a single polynomial term that is $a = bx^n$. From the physical point of view, the coefficient depends on the velocity gradient. This relationship is referred to as power relationship and it can be written as

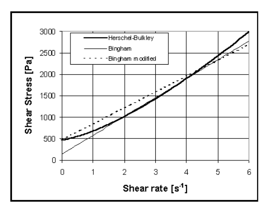



*Fig. -1.7. The shear stress as a function of the shear rate.*

$$\tau = \overbrace{K \left(\frac{dU}{dx}\right)^{n-1}}^{viscosity} \left(\frac{dU}{dx}\right) \tag{1.13}$$

The new coefficients $(n, K)$ in equation (1.13) are constant. When $n = 1$ equation represent Newtonian fluid and $K$ becomes the familiar $\mu$. The viscosity coefficient is always positive. When $n$, is above one, the liquid is dilettante. When $n$ is below one, the fluid is pseudoplastic. The liquids which satisfy equation (1.13) are referred to as purely viscous fluids. Many fluids satisfy the above equation. Fluids that show increase in the viscosity (with increase of the shear) referred to as thixotropic and those that show decrease are called reopectic fluids (see Figure 1.5).

Materials which behave up to a certain shear stress as a solid and above it as a liquid are referred as Bingham liquids. In the simple case, the "liquid side" is like Newtonian fluid for large shear stress. The general relationship for simple Bingham flow is

$$\tau_{xy} = -\mu \ \pm \tau_0 \qquad if\,|\tau_{yx}| > \tau_0 \tag{1.14}$$

$$\frac{dU_x}{dy} = 0 \qquad if\,|\tau_{yx}| < \tau_0 \tag{1.15}$$

There are materials that simple Bingham model does not provide dequate explanation and a more sophisticate model is required. The Newtonian part of the model has to be replaced by power liquid. For example, according to Ferraris at el[1] concrete behaves as shown in Figure 1.7. However, for most practical purposes, this kind of figures isn't used in regular engineering practice.

### 1.5.3 Kinematic Viscosity

The kinematic viscosity is another way to look at the viscosity. The reason for this new definition is that some experimental data are given in this form. These results also explained better using the new definition. The kinematic viscosity embraces both the viscosity and density properties of a fluid. The above equation shows that the dimensions of $\nu$ to be square meter per second, $[m^2/sec]$, which are acceleration units (a combination of kinematic terms). This fact explains the name "kinematic" viscosity. The kinematic viscosity is defined as

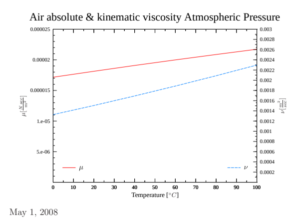



*Fig. -1.8. Air viscosity as a function of the temperature.*

$$\nu = \frac{\mu}{\rho} \tag{1.16}$$

The gas density decreases with the temperature. However, The increase of the absolute viscosity with the temperature is enough to overcome the increase of density and thus, the kinematic viscosity also increase with the temperature for many materials.

[1] C. Ferraris, F. de Larrard and N. Martys, Materials Science of Concrete VI, S. Mindess and J. Skalny, eds., 215-241 (2001)

### 1.5.4 Estimation of The Viscosity

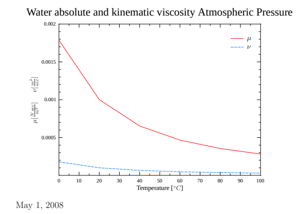



*Fig. -1.9. Water viscosity as a function temperature.*

The absolute viscosity of many fluids relatively doesn't change with the pressure but very sensitive to temperature. For isothermal flow, the viscosity can be considered constant in many cases. The variations of air and water as a function of the temperature at atmospheric pressure are plotted in Figures 1.8 and 1.9.

Some common materials (pure and mixture) have expressions that provide an estimate. For many gases, Sutherland's equation is used and according to the literature, provides reasonable results[2] for the range of $-40°C$ to $1600°C$

$$\mu = \mu_0 \, \frac{0.555 \, T_{i0} + Suth}{0.555 \, T_{in} + Suth} \left(\frac{T}{T_0}\right)^{\frac{3}{2}} \tag{1.17}$$

Where

| | |
|---|---|
| $\mu$ | viscosity at input temperature T |
| $\mu_0$ | reference viscosity at reference temperature, $T_{i0}$ |
| $T_{in}$ | input temperature in degrees Kelvin |
| $T_{i0}$ | reference temperature in degrees Kelvin |
| $Suth$ | Suth is Sutherland's constant and it is presented in the Table 1.1 |

.

Example 1.3:
*Calculate the viscosity of air at 800K based on Sutherland's equation. Use the data provide in Table 1.1.*

SOLUTION

Applying the constants from Suthelnd's table provides

$$\mu = 0.00001827 \times \frac{0.555 \times 524.07 + 120}{0.555 \times 800 + 120} \times \left(\frac{800}{524.07}\right)^{\frac{3}{2}} \sim 2.51\,10^{-5} \left[\frac{N\,sec}{m^2}\right]$$

The viscosity increases almost by 40%. The observed viscosity is about $\sim 3.710^{-5}$.

End Solution

#### Liquid Metals

[2]This author is ambivalent about statement.

| coefficients / Material | chemical formula | Sutherland | $T_{iO}[K]$ | $\mu_0(N\,sec/m^2)$ |
|---|---|---|---|---|
| ammonia | $NH_3$ | 370 | 527.67 | 0.00000982 |
| standard air | | 120 | 524.07 | 0.00001827 |
| carbon dioxide | $CO_2$ | 240 | 527.67 | 0.00001480 |
| carbon monoxide | $CO$ | 118 | 518.67 | 0.00001720 |
| hydrogen | $H_2$ | 72 | 528.93 | 0.0000876 |
| nitrogen | $N_2$ | 111 | 540.99 | 0.0001781 |
| oxygen | $O_2$ | 127 | 526.05 | 0.0002018 |
| sulfur dioxide | $SO_2$ | 416 | 528.57 | 0.0001254 |

*Table -1.1. The list for Sutherland's equation coefficients for selected materials.*

| Substance | formula | Temperature $T\,[^\circ C]$ | Viscosity $[\frac{N\,sec}{m^2}]$ |
|---|---|---|---|
| | $i-C_4\,H_{10}$ | 23 | 0.0000076 |
| | $CH_4$ | 20 | 0.0000109 |
| | $CO_2$ | 20 | 0.0000146 |
| oxygen | $O_2$ | 20 | 0.0000203 |
| mercury vapor | $Hg$ | 380 | 0.0000654 |

*Table -1.2. Viscosity of selected gases.*

| Substance | formula | Temperature $T\ [^\circ C]$ | Viscosity $[\frac{N\,sec}{m^2}]$ |
|---|---|---|---|
| | $(C_2H_5)O$ | 20 | 0.000245 |
| | $C_6H_6$ | 20 | 0.000647 |
| | $Br_2$ | 26 | 0.000946 |
| | $C_2H_5OH$ | 20 | 0.001194 |
| | $Hg$ | 25 | 0.001547 |
| | $H_2SO_4$ | 25 | 0.01915 |
| Olive Oil | | 25 | 0.084 |
| Castor Oil | | 25 | 0.986 |
| Clucuse | | 25 | 5-20 |
| Corn Oil | | 20 | 0.072 |
| SAE 30 | | - | 0.15-0.200 |
| SAE 50 | | $\sim 25^\circ C$ | 0.54 |
| SAE 70 | | $\sim 25^\circ C$ | 1.6 |
| Ketchup | | $\sim 20^\circ C$ | 0,05 |
| Ketchup | | $\sim 25^\circ C$ | 0,098 |
| Benzene | | $\sim 20^\circ C$ | 0.000652 |
| Firm glass | | - | $\sim 1 \times 10^7$ |
| Glycerol | | 20 | 1.069 |

*Table -1.3. Viscosity of selected liquids.*

| chemical component | Molecular Weight | $T_c$[K] | $P_c$[Bar] | $\mu_c$ $[\frac{N\,sec}{m^2}]$ |
|---|---|---|---|---|
| $H_2$ | 2.016 | 33.3 | 12.9696 | 3.47 |
| $He$ | 4.003 | 5.26 | 2.289945 | 2.54 |
| $Ne$ | 20.183 | 44.5 | 27.256425 | 15.6 |
| $Ar$ | 39.944 | 151 | 48.636 | 26.4 |
| $Xe$ | 131.3 | 289.8 | 58.7685 | 49. |
| Air "mix" | 28.97 | 132 | 36.8823 | 19.3 |
| $CO_2$ | 44.01 | 304.2 | 73.865925 | 19.0 |
| $O_2$ | 32.00 | 154.4 | 50.358525 | 18.0 |
| $C_2H_6$ | 30.07 | 305.4 | 48.83865 | 21.0 |
| $CH_4$ | 16.04 | 190.7 | 46.40685 | 15.9 |
| Water | | 647.096 K | 22.064 [MPa] | |

*Table -1.4. The properties at the critical stage and their values of selected materials.*

Liquid metal can be considered as a Newtonian fluid for many applications. Furthermore, many aluminum alloys are behaving as a Newtonian liquid until the first solidification appears (assuming steady state thermodynamics properties). Even when there is a solidification (mushy zone), the metal behavior can be estimated as a Newtonian material (further reading can be done in this author's book "Fundamentals of Die Casting Design"). Figure 1.10 exhibits several liquid metals (from The Reactor Handbook, Vol. Atomic Energy Commission AECD-3646 U.S. Government Printing Office, Washington D.C. May 1995 p. 258.)

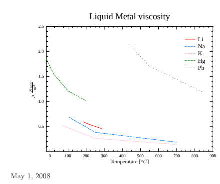



*Fig. -1.10. Liquid metals viscosity as a function of the temperature.*

### The General Viscosity Graphs

In case "ordinary" fluids where information is limit, Hougen et al suggested to use graph similar to compressibility chart. In this graph, if one point is well documented, other points can be estimated. Furthermore, this graph also shows the trends. In Figure 1.11 the relative viscosity $\mu_r = \mu/\mu_c$ is plotted as a function of relative temperature, $T_r$. $\mu_c$ is the viscosity at critical condition and $\mu$ is the viscosity at any given condition. The lines of constant relative pressure, $P_r = P/P_c$ are drawn. The lower pressure is, for practical purpose, $\sim 1[bar]$.

The critical pressure can be evaluated in the following three ways. The simplest way is by obtaining the data from Table 1.4 or similar information. The second way, if the information is available and is close enough to the critical point, then the critical viscosity is obtained as

$$\mu_c = \frac{\overbrace{\mu}^{given}}{\underbrace{\mu_r}_{\text{figure 1.11}}} \tag{1.18}$$

The third way, when none is available, is by utilizing the following approximation

$$\mu_c = \sqrt{M T_c} \tilde{v_c}^{2/3} \tag{1.19}$$

Where $\tilde{v_c}$ is the critical molecular volume and $M$ is molecular weight. Or

$$\mu_c = \sqrt{M} {P_c}^{2/3} {T_c}^{-1/6} \tag{1.20}$$

Calculate the reduced pressure and the reduced temperature and from the Figure 1.11 obtain the reduced viscosity.

Example 1.4:
*Estimate the viscosity of oxygen, $O_2$ at $100°C$ and 20[Bar].*

<u>Solution</u>

The critical condition of oxygen are $P_c = 50.35[Bar]$ $T_c = 154.4$ $\mu_c = 18 \left[\frac{N\,sec}{m^2}\right]$ The value of the reduced temperature is

$$T_r \sim \frac{373.15}{154.4} \sim 2.41$$

The value of the reduced pressure is

$$P_r \sim \frac{20}{50.35} \sim 0.4$$

From Figure 1.11 it can be obtained $\mu_r \sim 1.2$ and the predicted viscosity is

$$\mu = \mu_c \overbrace{\left(\frac{\mu}{\mu_c}\right)}^{Table} = 18 \times 1.2 = 21.6[Nsec/m^2]$$

The observed value is 24[N sec/$m^2$][3].

End Solution

**Viscosity of Mixtures**

In general the viscosity of liquid mixture has to be evaluated experimentally. Even for homogeneous mixture, there isn't silver bullet to estimate the viscosity. In this book, only the mixture of low density gases is discussed for analytical expression. For most

[3]Kyama, Makita, Rev. Physical Chemistry Japan Vol. 26 No. 2 1956.

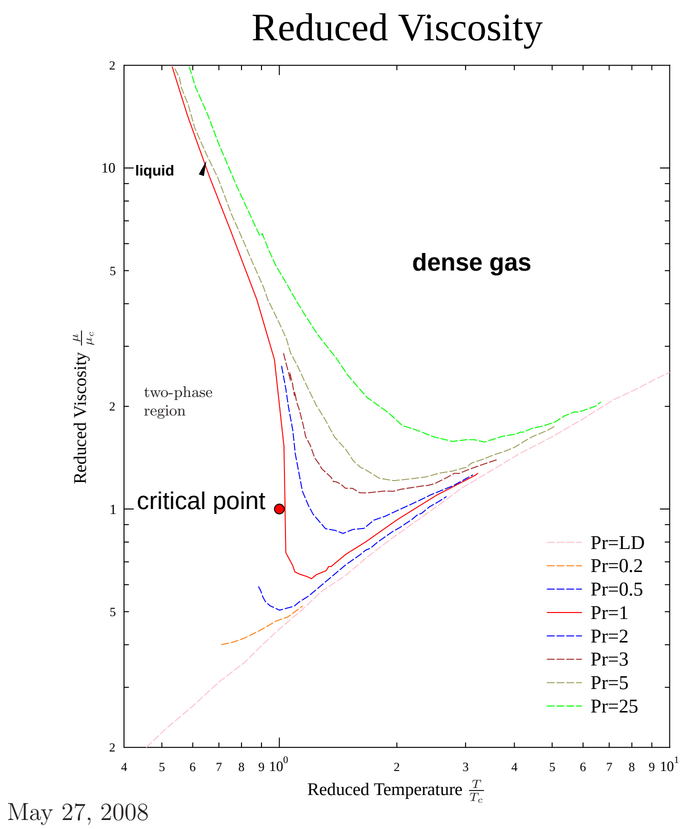



*Fig. -1.11. Reduced viscosity as function of the reduced temperature.*

cases, the following Wilke's correlation for gas at low density provides a result in a reasonable range.

$$\mu_{mix} = \sum_{i=1}^{n} \frac{x_i \, \mu_i}{\sum_{j=1}^{n} x_i \, \Phi_{ij}} \tag{1.21}$$

where $\Phi_i j$ is defined as

$$\Phi_{ij} = \frac{1}{\sqrt{8}} \sqrt{1 + \frac{M_i}{M_j}} \left( 1 + \sqrt{\frac{\mu_i}{\mu_j}} \sqrt[4]{\frac{M_j}{M_i}} \right)^2 \tag{1.22}$$

Here, $n$ is the number of the chemical components in the mixture. $x_i$ is the mole fraction of component $i$, and $\mu_i$ is the viscosity of component $i$. The subscript $i$ should be used for the $j$ index. The dimensionless parameter $\Phi_{ij}$ is equal to one when $i = j$. The mixture viscosity is highly nonlinear function of the fractions of the components.

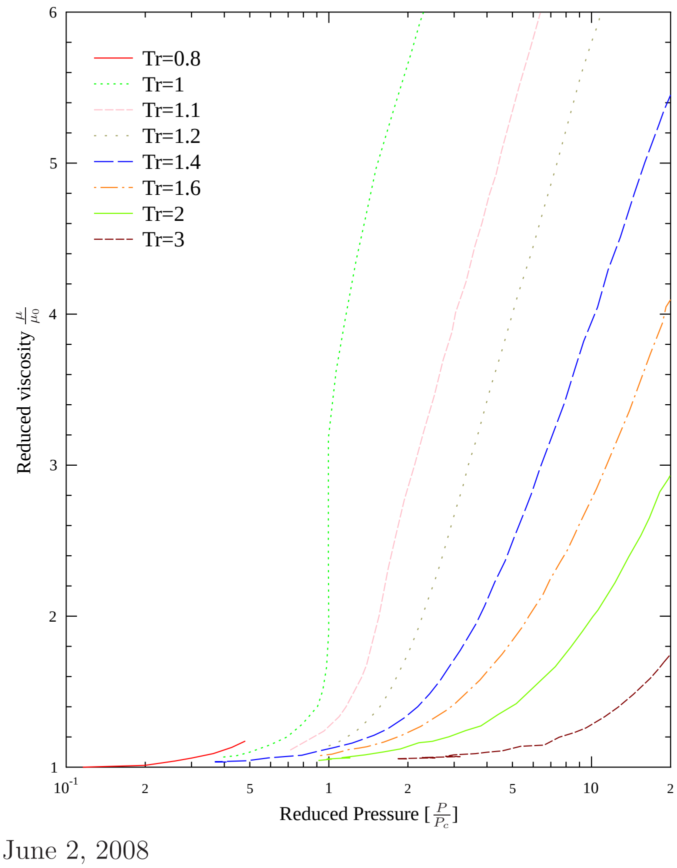



*Fig. -1.12. Reduced viscosity as function of the reduced temperature.*

Example 1.5:

*Calculate the viscosity of a mixture (air) made of 20% oxygen, $O_2$ and 80% nitrogen $N_2$ for the temperature of $20°C$.*

SOLUTION

The following table summarize the known details

| i | Component | Molecular Weight, $M$ | Mole Fraction, $x$ | Viscosity, $\mu$ |
|---|---|---|---|---|
| 1 | $O_2$ | 32. | 0.2 | 0.0000203 |
| 2 | $N_2$ | 28. | 0.8 | 0.00001754 |

| i | j | $M_i/M_j$ | $\mu_i/\mu_j$ | $\Phi_{ij}$ |
|---|---|---|---|---|
| 1 | 1 | 1.0 | 1.0 | 1.0 |
| | 2 | 1.143 | 1.157 | 1.0024 |
| 2 | 1 | 0.875 | .86 | 0.996 |
| | 2 | 1.0 | 1.0 | 1.0 |

$$\mu_{mix} \sim \frac{0.2 \times 0.0000203}{0.2 \times 1.0 + 0.8 \times 1.0024} + \frac{0.8 \times 0.00001754}{0.2 \times 0.996 + 0.8 \times 1.0} \sim 0.0000181 \left[\frac{N\, sec}{m^2}\right]$$

The observed value is $\sim 0.0000182 \left[\frac{N\, sec}{m^2}\right]$.

End Solution

In very low pressure, in theory, the viscosity is only a function of the temperature with a "simple" molecular structure. For gases with very long molecular structure or complexity structure these formulas cannot be applied. For some mixtures of two liquids it was observed that at a low shear stress, the viscosity is dominated by a liquid with high viscosity and at high shear stress to be dominated by a liquid with the low viscosity liquid. The higher viscosity is more dominate at low shear stress. Reiner and Phillippoff suggested the following formula

$$\frac{dU_x}{dy} = \left( \frac{1}{\mu_\infty + \dfrac{\mu_0 - \mu_\infty}{1 + \left(\frac{\tau_{xy}}{\tau_s}\right)^2}} \right) \tau_{xy} \tag{1.23}$$

Where the term $\mu_\infty$ is the experimental value at high shear stress. The term $\mu_0$ is the experimental viscosity at shear stress approaching zero. The term $\tau_s$ is the characteristic shear stress of the mixture. An example for values for this formula, for Molten Sulfur at temperature $120°C$ are $\mu_\infty = 0.0215 \left(\frac{N\, sec}{m^2}\right)$, $\mu_0 = 0.00105 \left(\frac{N\, sec}{m^2}\right)$, and $\tau_s = 0.0000073 \left(\frac{kN}{m^2}\right)$. This equation (1.23) provides reasonable value only up to $\tau = 0.001 \left(\frac{kN}{m^2}\right)$.

Figure 1.12 can be used for a crude estimate of dense gases mixture. To estimate the viscosity of the mixture with $n$ component Hougen and Watson's method for pseudocritial properties is adapted. It this method the following is defined as

$$P_{c_{mix}} = \sum_{i=1}^{n} x_i P_{c_i} \tag{1.24}$$

$$T_{c_{mix}} = \sum_{i=1}^{n} x_i T_{c_i} \tag{1.25}$$

and

$$\mu_{c_{mix}} = \sum_{i=1}^{n} x_i \mu_{c_i} \tag{1.26}$$

Example 1.6:

*An inside cylinder with a radius of 0.1 [m] rotates concentrically within a fixed cylinder of 0.101 [m] radius and the cylinders length is 0.2 [m]. It is given that a moment of 1 $[N \times m]$ is required to maintain an angular velocity of 31.4 revolution per second. Estimate the liquid viscosity used between the cylinders.*

<u>Solution</u>

The moment or the torque is transmitted through the liquid to the outer cylinder. Control volume around the inner cylinder shows that moment is a function of the area and shear stress. The shear stress calculations can be estimated as a linear between the two concentric cylinders. The velocity at the inner cylinders surface is

$$U_i = r\,\omega = 0.1 \times 1[rad/second] = 0.1[m/s] \tag{1.VI.a}$$

The velocity at the outer cylinder surface is zero. The velocity gradient may be assumed to be linear, hence,

$$\frac{dU}{dr} \cong \frac{0.1 - 0}{0.101 - 0.1} = 100 sec^{-1} \tag{1.VI.b}$$

The used moment is

$$M = \mu \frac{dU}{dr} = \overbrace{100}^{\frac{dU}{dr}} \times 2 \times 0.1 \times \pi \times 0.2 = \tag{1.VI.c}$$

End Solution

Example 1.7:

*A square block weighing 1.0 [kN] with a side surfaces area of 0.1 $[m^2]$ slides down an incline surface with an angle of 20° C. The surface is covered with oil film. The oil force a distance between the block and the inclined surface of $1 \times 10^{-6}[m]$ thick. What is the speed of the block at steady state? Assuming a linear velocity profile in the oil and that the whole oil is under steady state. The viscosity of the oil is $3 \times 10^{-5}[m^2/sec]$.*

<u>Solution</u>

The shear stress at the surface is estimated for steady state by

$$\tau = \mu \frac{dU}{dx} = 3 \times 10^{-5} \times \frac{U}{1 \times 10^{-6}} = 30\,U \tag{1.VII.a}$$

The total fiction force is then

$$f = \tau\, A = 0.1 \times 30, U = 3\, U \tag{1.VII.b}$$

The gravity force that act against the friction is equal to the friction hence

$$F_g = m\, g\, \sin\, 20^\circ = 3\, U \Longrightarrow U = \frac{m\, g\, \sin\, 20^\circ}{3} = \tag{1.VII.c}$$

End Solution

Example 1.8:

*Develop an expression for estimate of the torque required to rotate a disc in a narrow gap. The edge effects can be neglected. The gap is given and equal to $\delta$ and the rotation speed is $\omega$. The shear stress can be assumed to be linear.*

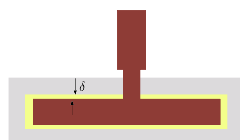


*Fig. -1.13. Rotating disc in a steady state.*

SOLUTION

In this cases the shear stress is a function of the radius, $r$ and expression has to be developed for it. In addition the differential area also increases and is a function of $r$. The shear stress can be estimated as

$$\tau \cong \mu \frac{U}{\delta} = \mu\, \frac{\omega\, r}{\delta} \tag{1.VIII.a}$$

This shear stress can be integrated for the entire area as

$$T = \int_0^R 2\, r\, \tau\, dA = 2 \int_0^R 2\, \mu\, r\, \frac{\omega\, r}{\delta}\, 2\, \pi\, r\, dr \tag{1.VIII.b}$$

The results of the integration is

$$F = \frac{\pi\, \mu\, \omega\, R^4}{\delta} \tag{1.VIII.c}$$

End Solution

## 1.6 Fluid Properties

The fluids have many properties which are similar to solid. A discussion of viscosity and surface tension should be part of this section but because special importance these topics have a separate sections. The rest of the properties lumped into this section.

### 1.6.1 Fluid Density

The density is a property that is simple to analyzed and understand. Examples to described usage of property are provided.

Example 1.9:
*A steel tank filled with water undergoes heating from 27° C to 127° C. The initial pressure can be assumed to atmospheric. Due to the change temperature the tank (the steel) undergoes linear expansion of* $8\,10^{-6} per\,^{\circ}\mathrm{C}$*. State your assumptions.*

Solution

The expansion of the steel tank will be due to two contributions: one from the thermal Expansion and two pressure increase in the tank. For this example, it is assumed that the expansion due to pressure increase is negligible. The tank volume change under the assumptions the tank walls remain straight is

$$V_2 = V_1 \overbrace{(1+\alpha\Delta\, T)^3}^{\text{thermal expansion}} \tag{1.IX.a}$$

The more accurate calculations require looking into the steam tables. As approximation the relationship between the pressure and density in the liquid phase as

$$\frac{\rho_2}{\rho_1} = \frac{1}{1-\dfrac{P_2-P_1}{E}} = \frac{E}{E-\Delta P} \tag{1.IX.b}$$

where $E$ denotes the modulus of elasticity for the water $2.15\,10^9 (N/m^2)$ The water mass in the tank remain constant $m_1 = m_2 \longrightarrow \rho_1\, V_1 = \rho_2\, V_2$. The change of density is reversed of the change of volume.

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{E}{E-\Delta P} \tag{1.IX.c}$$

or using equation (1.IX.a)

$$(1+\alpha\Delta\, T)^3 = \frac{E-\Delta P}{E} \tag{1.IX.d}$$

or expanding the cubical equation and neglecting high power terms of $\alpha$.

$$E\,(1+\alpha\Delta\, T)^3 - E = P_2 - P_1 \Longrightarrow P_2 \sim P_1 + (3\,\alpha + \cdots)E \tag{1.IX.e}$$

$$P_1 = 3 \times 8\,10^{-6} \times 100 \times 2.15\,10^9 =$$

End Solution

### 1.6.2 Bulk Modulus

Similar to solids (hook's law), liquids have a property that describes the volume change as results of pressure change for constant temperature. It can be noted that this property is not the result of the equation of state but related to it. Bulk modulus is usually obtained from experimental or theoretical or semi theoretical (theory with experimental work) to fit energy–volume data. Most (theoretical) studies are obtained by uniformly changing the unit cells in global energy variations especially for isotropic systems ( where the molecules has a structure with cubic symmetries). The bulk modulus is defined as

$$B_T = -v \left(\frac{\partial P}{\partial v}\right)_T \tag{1.27}$$

Using the identity of $v = 1/\rho$ transfers equation (1.27) into

$$B_T = \rho \left(\frac{\partial P}{\partial \rho}\right)_T \tag{1.28}$$

The bulk modulus for several liquids is presented in Table 1.5.

*Table -1.5. The bulk modulus for selected material with the critical temperature and pressure $na \longrightarrow$ not available and $nf \longrightarrow$ not found (exist but was not found in the literature).*

| chemical component | Bulk Modulus $10^9 \frac{N}{m}$ | $T_c$ | $P_c$ |
|---|---|---|---|
| Acetic Acid | 2.49 | 593K | 57.8 [Bar] |
| Acetone | 0.80 | 508 K | 48 [Bar] |
| Benzene | 1.10 | 562 K | 4.74 [MPa] |
| Carbon Tetrachloride | 1.32 | 556.4 K | 4.49 [MPa] |
| Ethyl Alcohol | 1.06 | 514 K | 6.3 [Mpa] |
| Gasoline | 1.3 | nf | nf |
| Glycerol | 4.03-4.52 | 850 K | 7.5 [Bar] |
| Mercury | 26.2-28.5 | 1750 K | 172.00 [MPa] |
| Methyl Alcohol | 0.97 | Est 513 | Est 78.5 [Bar] |
| Nitrobenzene | 2.20 | nf | nf |
| Olive Oil | 1.60 | nf | nf |
| Paraffin Oil | 1.62 | nf | nf |
| SAE 30 Oil | 1.5 | na | na |
| Seawater | 2.34 | na | na |
| Toluene | 1.09 | 591.79 K | 4.109 [MPa] |
| Turpentine | 1.28 | na | na |
| Water | 2.15-2.174 | 647.096 K | 22.064 [MPa] |

In the literature, additional expansions for similar parameters are defined. The thermal expansion is defined as

$$\beta_P = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P \tag{1.29}$$

This parameter indicates the change of volume due to temperature change when the pressure is constant. Another definition is referred as coefficient of tension and it is defined as

$$\beta_v = \frac{1}{P}\left(\frac{\partial P}{\partial T}\right)_v \tag{1.30}$$

This parameter indicates the change of the pressure due to the change of temperature (where $v = constant$). These definitions are related to each other. This relationship is obtained by the observation that the pressure as a function of the temperature and specific volume as

$$P = f(T, v) \tag{1.31}$$

The full pressure derivative is

$$dP = \left(\frac{\partial P}{\partial T}\right)_v dT + \left(\frac{\partial P}{\partial v}\right)_T dv \tag{1.32}$$

On constant pressure lines, $dP = 0$, and therefore equation (1.32) is

$$0 = \left(\frac{\partial P}{\partial T}\right)_v dT + \left(\frac{\partial P}{\partial v}\right)_T dv \tag{1.33}$$

From equation (1.33) follows that

$$\left.\frac{dv}{dT}\right|_{P=const} = -\frac{\left(\frac{\partial P}{\partial T}\right)_v}{\left(\frac{\partial P}{\partial v}\right)_T} \tag{1.34}$$

Equation (1.34) indicates that relationship for these three coefficients is

$$\beta_T = -\frac{\beta_v}{\beta_P} \tag{1.35}$$

The last equation (1.35) sometimes is used in measurement of the bulk modulus.

The increase of the pressure increases the bulk modulus due to the molecules increase of the rejecting forces between each other when they are closer. In contrast, the temperature increase results in reduction of the bulk of modulus because the molecular are further away.

Example 1.10:
*Calculate the modulus of liquid elasticity that reduced 0.035 per cent of its volume by applying a pressure of 5[Bar].*

SOLUTION

Using the definition for the bulk modulus

$$\beta_T = -v\frac{\partial P}{\partial v} \sim \frac{v}{\Delta v}\Delta P = \frac{5}{0.00035} \sim 14285.714[Bar]$$

End Solution

Example 1.11:
*Calculate the pressure needed to apply on water to reduce its volume by 1 per cent. Assume the temperature to be $20°C$.*

SOLUTION

Using the definition for the bulk modulus

$$\Delta P \sim \beta_T \frac{\Delta v}{v} \sim 2.15\,10^9 .01 = 2.15\,10^7 [N/m^2] = 215[Bar]$$

End Solution

Example 1.12:

*Two layers of two different liquids are contained in a very solid tank. Initially the pressure in the tank is $P_0$. The liquids are compressed due to the pressure increases. The new pressure is $P_1$. The area of the tank is $A$ and liquid A height is $h_1$ and liquid B height is $h_2$. Estimate the change of the heights of the liquids depicted in the Figure 1.14. State your assumptions.*

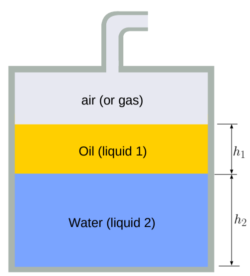



*Fig. -1.14. Two liquid layers under pressure.*

SOLUTION

The volume change in a liquid is

$$B_T \cong \frac{\Delta P}{\Delta V/V} \tag{1.XII.a}$$

Hence the change for the any liquid is

$$\Delta h = \frac{\Delta P}{A\,B_T/V} = \frac{h\,\Delta P}{B_T} \tag{1.XII.b}$$

The total change when the hydrostatic pressure is ignored.

$$\Delta h_{1+2} = \Delta P \left( \frac{h_1}{B_{T1}} + \frac{h_2}{B_{T2}} \right) \tag{1.XII.c}$$

End Solution

Example 1.13:

*In internet the following problem (whith latex modification) was posted which related to Pushka equation.*

*A cylindrical steel pressure vessel with volume 1.31 $m^3$ is to be tested. The vessel is entirely filled with water, then a piston at one end of the cylinder is pushed in until the pressure inside the vessel has increased by 1000 kPa. Suddenly, a safety plug on the top bursts. How many liters of water come out?*

*Relevant equations and data suggested by the user were:* $B_T = 0.2x10^10N/m^2$, $P_1 = P_0 + \rho\, g\, h$, $P_1 = -B_T \Delta V/V$

*with the suggested solution of*

*I am assuming that I have to look for $\Delta V$ as that would be the water that comes out causing the change in volume.*

$$\Delta V = -V(\Delta P)/B_T = -1.31(1000)/(0.2x10^10)\Delta V = 6.55 * 10^-7$$

*Another user suggest that:*

*We are supposed to use the bulk modulus from our textbook, and that one is $0.2*10^10$. Anything else would give a wrong answer in the system. So with this bulk modulus, is 0.655L right?*

*In this post several assumptions were made. What is the correct way to solve this problem.*

Example 1.14:

*The hydrostatic pressure was neglected in example 1.12. In some places the ocean deepth is many kilometers (the deepest places is more than 10 kilometers). For this example, calculate the density change in the bottom of 10 kilometers using two methods. In one method assume that the density is remain constant until the bottom. In the second method assume that the density is a function of the pressure.*

Solution

For the the first method the density is

$$B_T \cong \frac{\Delta P}{\Delta V/V} \Longrightarrow \Delta V = V \frac{\Delta P}{B_T} \tag{1.XIV.a}$$

The density at the surface is $\rho = m/V$ and the density at point $x$ from the surface the density is

$$\rho(x) = \frac{m}{V - \Delta V} \Longrightarrow \rho(x) = \frac{m}{V - V\dfrac{\Delta P}{B_T}} \tag{1.XIV.b}$$

In the Chapter on static it will be shown that the change pressure is

$$\Delta P = g \int_0^x \rho(x) dx \tag{1.XIV.c}$$

Combining equation (1.XIV.b) with equation (1.XIV.c) yields

$$\rho(x) = \frac{m}{V - V\dfrac{g\displaystyle\int_0^x \rho(x) dx}{B_T}} \tag{1.XIV.d}$$

Equation can be rearranged to be

$$\rho(x) = \frac{m}{V\left(1 - \dfrac{g}{B_T}\displaystyle\int_0^x \rho(x) dx\right)} \Longrightarrow \rho(x) = \frac{\rho_0}{\left(1 - \dfrac{g}{B_T}\displaystyle\int_0^x \rho(x) dx\right)} \tag{1.XIV.e}$$

Equation (1.XIV.e) is an integral equation which is discussed in the appendix[4]. . It is convenient to change further equation (1.XIV.e) to

$$1 - \frac{g}{B_T}\int_0^x \rho(x) dx = \frac{\rho_0}{\rho(x)} \tag{1.XIV.f}$$

The integral equation (1.XIV.f) can be converted to differential equation when the two sides under differentiation

$$\frac{g}{B_T}\rho(x) + \frac{\rho_0}{\rho(x)^2}\frac{d\,\rho(x)}{dx} = 0 \tag{1.XIV.g}$$

or

$$\frac{g\,\rho(x)^3}{B_T\,\rho_0} + \frac{d\,\rho(x)}{dx} = 0 \tag{1.XIV.h}$$

The solution is

$$\frac{\rho_0\,B_T}{2\,g\,\rho^2} = x + c \tag{1.XIV.i}$$

or rearranged as

$$\rho = \sqrt{\frac{\rho_0\,B_T}{2\,g\,(x+c)}} \tag{1.XIV.j}$$

[4]Under construction

The integration constant can be found by the fact that the density at the $x = 0$ is $\rho_0$

$$\rho_0 = \sqrt{\frac{\rho_0\, B_T}{2\, g\, (c)}} \Longrightarrow c = \frac{B_T}{2\, g\, \rho_0} \tag{1.XIV.k}$$

Hence the solution is

$$\frac{\rho}{\rho_0} = \sqrt{\frac{\rho_0\, B_T}{2\, g\, \rho_0\, x + B_T}} \tag{1.XIV.l}$$

In the "constant" density approach, the density at the bottom using equation (1.XIV.e) is

$$\rho = \frac{\rho_0}{1 - \dfrac{g}{B_T} g\, \rho_0\, x} \Longrightarrow \frac{\rho_0\, B_T}{B_T - g\, \rho_0\, x} \tag{1.XIV.m}$$

End Solution

*Advance material can be skipped*

Example 1.15:

*Water in deep sea undergoes compresion due to hydrostic pressure. That is the density is function of the depth. For constant bulk modulus, it was shown in "Fundamentals of Compressible Flow" by this author that the speed of sound is*

$$c = \sqrt{\frac{B_T}{\rho}} \tag{1.XV.a}$$

*Calculate the time it take for a sound wave to propogate from the surface to a depth $D$ penpendicular the surface. Assume that no variation of the temperatuere. For the purpose of this excerss, the salinity can be complity ignored.*

Solution

The equation for the sound speed is taken here as correct for very local point. However, the desnitsy is different for evry point since the density varies and the desnity is a function of the depth. The speed of sound at any depth point, $x$, is

$$c = \sqrt{\frac{B_T}{\dfrac{\rho_0\, B_T}{B_T - g\, \rho_0\, x}}} = \sqrt{\frac{B_T - g\, \rho_0\, x}{\rho_0}} \tag{1.XV.b}$$

The time the sound travel a small intervel distance, $dx$ is

$$d\tau = \frac{dx}{\sqrt{\dfrac{B_T - g\, \rho_0\, x}{\rho_0}}} \tag{1.XV.c}$$

The time takes for the sound the travel the whole distance is the integration of infinitesimal time

$$t = \int_0^D \frac{dx}{\sqrt{\dfrac{B_T - g\,\rho_0\,x}{\rho_0}}} \tag{1.XV.d}$$

The solution of equation (1.XV.d) is

$$t = \sqrt{\rho_0}\,\left(2\,\sqrt{B_T} - 2\,\sqrt{B_T - D}\right) \tag{1.XV.e}$$

The time to travel according to the standard procedure is

$$t = \frac{D}{\sqrt{\dfrac{B_T}{\rho_0}}} = \frac{D\,\sqrt{\rho_0}}{\sqrt{B_T}} \tag{1.XV.f}$$

The ratio between the corrected estimated to the standard caclulation is

$$\text{correction ratio } = \frac{\sqrt{\rho_0}\,\left(2\,\sqrt{B_T} - 2\,\sqrt{B_T - D}\right)}{\dfrac{D\,\sqrt{\rho_0}}{\sqrt{B_T}}} \tag{1.XV.g}$$

End Solution

#### 1.6.2.1 Bulk Modulus of Mixtures

In the discussion above it was assumed that the liquid is pure. In this short section a discussion about the bulk modulus averaged is presented. When more than one liquid are exposed to pressure the value of these two (or more liquids) can have to be added in special way. The definition of the bulk modulus is given by equation (1.27) or (1.28) and can be written (where the partial derivative can looks as delta $\Delta$ as

$$\partial V = \frac{V\,\partial P}{B_T} \cong \frac{V\,\Delta P}{B_T} \tag{1.36}$$

The total change is compromised by the change of individual liquids or phases if two materials are present. Even in some cases of emulsion (a suspension of small globules of one liquid in a second liquid with which the first will not mix) the total change is the summation of the individuals change. In case the total change isn't, in special mixture, another approach with taking into account the energy-volume is needed. Thus, the total change is

$$\partial V = \partial V_1 + \partial V_2 + \cdots \partial V_i \cong \Delta V_1 + \Delta V_2 + \cdots \Delta V_i \tag{1.37}$$

Substituting equation (1.36) into equation (1.37) results in

$$\partial V = \frac{V_1\,\partial P}{B_{T1}} + \frac{V_2\,\partial P}{B_{T2}} + \cdots + \frac{V_i\,\partial P}{B_{Ti}} \cong \frac{V_1\,\Delta P}{B_{T1}} + \frac{V_2\,\Delta P}{B_{T2}} + \cdots + \frac{V_i\,\Delta P}{B_{Ti}} \tag{1.38}$$

Under the main assumption in this model the total volume is comprised of the individual volume hence,

$$V = x_1\, V + x_1\, V + \cdots + x_i\, V \tag{1.39}$$

Where $x_1$, $x_2$ and $x_i$ are the fraction volume such as $x_i = V_i/V$. Hence, using this identity and the fact that the pressure is change for all the phase uniformly equation (1.39) can be written as

$$\partial V = V\,\partial P\left(\frac{x_1}{B_{T1}} + \frac{x_2}{B_{T2}} + \cdots + \frac{x_i}{B_{Ti}}\right) \cong V\,\Delta P\left(\frac{x_1}{B_{T1}} + \frac{x_2}{B_{T2}} + \cdots + \frac{x_i}{B_{Ti}}\right) \tag{1.40}$$

Rearranging equation (1.40) yields

$$v\,\frac{\partial P}{\partial v} \cong v\,\frac{\Delta P}{\Delta v} = \frac{1}{\left(\dfrac{x_1}{B_{T1}} + \dfrac{x_2}{B_{T2}} + \cdots + \dfrac{x_i}{B_{Ti}}\right)} \tag{1.41}$$

Equation (1.41) suggested an averaged new bulk modulus

$$B_{T\,mix} = \frac{1}{\left(\dfrac{x_1}{B_{T1}} + \dfrac{x_2}{B_{T2}} + \cdots + \dfrac{x_i}{B_{Ti}}\right)} \tag{1.42}$$

In that case the equation for mixture can be written as

$$v\,\frac{\partial P}{\partial v} = B_{T\,mix} \tag{1.43}$$

*End Advance material*

## 1.7 Surface Tension

The surface tension manifested itself by a rise or depression of the liquid at the free surface edge. Surface tension is also responsible for the creation of the drops and bubbles. It also responsible for the breakage of a liquid jet into other medium/phase to many drops (atomization). The surface tension is force per length and is measured by [N/m] and is acting to stretch the surface.

Surface tension results from a sharp change in the density between two adjoined phases or materials. There is a common misconception for the source of the surface tension. In many (physics, surface tension, and fluid mechanics) books explained that the surface tension is a result from unbalance molecular cohesive forces. This explanation is wrong since it is in conflict with Newton's second law (see example ?). This erroneous explanation can be traced to Adam's book but earlier source may be found.

The relationship between the surface tension and the pressure on the two sides of the surface is based on geometry. Consider a small element of surface. The pressure on one side is $P_i$ and the pressure on the other side is $P_o$. When the surface tension is constant, the horizontal forces cancel each other because symmetry. In the vertical direction, the surface tension forces are pulling the surface upward. Thus, the pressure difference has to balance the surface tension. The forces in the vertical direction reads

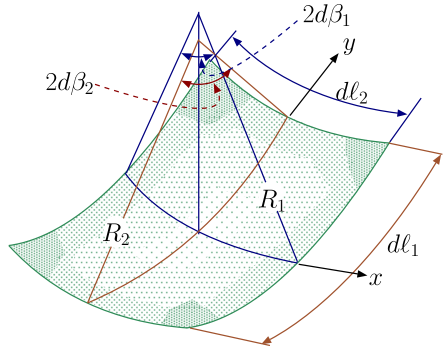



*Fig. -1.15. Surface tension control volume analysis describing principals radii.*

$$(P_i - P_o)\, d\ell_1\, d\ell_2 = \Delta\, P d\ell_1\, d\ell_2 = 2\,\sigma d\ell_1 \sin\beta_1 + 2\,\sigma d\ell_2 \sin\beta_2 \tag{1.44}$$

For a very small area, the angles are very small and thus $(\sin\beta \sim \beta)$. Furthermore, it can be noticed that $d\ell_i \sim 2\,R_i\, d\beta_i$. Thus, the equation (1.44) can be simplified as

$$\Delta\, P = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{1.45}$$

Equation (1.45) predicts that pressure difference increase with inverse of the radius. There are two extreme cases: one) radius of infinite and radius of finite size. The second with two equal radii. The first case is for an infinite long cylinder for which the equation (1.45) is reduced to

$$\Delta\, P = \sigma\left(\frac{1}{R}\right) \tag{1.46}$$

Other extreme is for a sphere for which the main radii are the same and equation (1.45) is reduced to

$$\Delta\, P = \frac{2\,\sigma}{R} \tag{1.47}$$

Where $R$ is the radius of the sphere. A soap bubble is made of two layers, inner and outer, thus the pressure inside the bubble is

$$\Delta\, P = \frac{4\,\sigma}{R} \tag{1.48}$$

Example 1.16:

*A glass tube is inserted into bath of mercury. It was observed that contact angle between the glass and mercury is 55° C.*

*The inner diameter is 0.02[m] and the outer diameter is 0.021[m]. Estimate the force due to the surface tension (tube is depicted in Figure 1.16). It can be assume that the contact angle is the same for the inside and outside part of the tube. Estimate the depression size. Assume that the surface tension for this combination of material is 0.5 [N/m]*

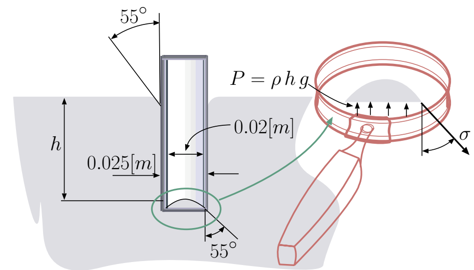



*Fig. -1.16. Glass tube inserted into mercury.*

<u>SOLUTION</u>

The mercury as free body that several forces act on it.

$$F = \sigma 2\,\pi \cos 55°\mathrm{C}\,(D_i + D_o) \tag{1.XVI.a}$$

This force is upward and the horizontal force almost canceled. However, if the inside and the outside diameters are considerable different the results is

$$F = \sigma 2\,\pi \sin 55°\mathrm{C}\,(D_o - D_o) \tag{1.XVI.b}$$

The balance of the forces on the meniscus show under the magnified glass are

$$P\,\overbrace{\pi\, r^2}^{A} = \sigma\, 2\,\pi\, r + \cancel{W}^{\sim 0} \tag{1.XVI.c}$$

or

$$g\,\rho\, h\,\pi\, r^2 = \sigma\, 2\,\pi\, r + \cancel{W}^{\sim 0} \tag{1.XVI.d}$$

Or after simplification

$$h = \frac{2\,\sigma}{g\,\rho\, r} \tag{1.XVI.e}$$

End Solution

Example 1.17:

*A Tank filled with liquid, which contains $n$ bubbles with equal radii, $r$. Calculate the minimum work required to increase the pressure in tank by $\Delta P$. Assume that the liquid bulk modulus is infinity.*

<u>SOLUTION</u>

The work is due to the change of the bubbles volume. The work is

$$w = \int_{r_0}^{r_f} \Delta P(v) dv \tag{1.49}$$

The minimum work will be for a reversible process. The reversible process requires very slow compression. It is worth noting that for very slow process, the temperature must remain constant due to heat transfer. The relationship between pressure difference and the radius is described by equation (1.47) for reversible process. Hence the work is

$$w = \int_{r_0}^{r_f} \overbrace{\frac{2\,\sigma}{r}}^{\Delta P} \overbrace{4\,\pi\,r^2\,dr}^{dv} = 8\,\pi\,\sigma \int_{r_0}^{r_f} r dr = 4\,\pi\,\sigma\left({r_f}^2 - {r_0}^2\right) \tag{1.50}$$

Where, $r_0$ is the radius at the initial stage and $r_f$ is the radius at the final stage.

The work for $n$ bubbles is then $4\,\pi\,\sigma\,n\left({r_f}^2 - {r_0}^2\right)$. It can be noticed that the work is negative, that is the work is done on the system.

End Solution

Example 1.18:

*Calcualte the rise of liquid between two dimentional parallel plates shown in Figure 1.17. Notice that previously a rise for circular tube was developed which different from simple one dimensional case. The distance between the two plates is $\ell$ and the and surface tension is $\sigma$. Assume that the contact angle is $0^{circ}$ (the maximum possible force). Cumpute the value for sufrace tension of $0.05[N/m]$, the density $1000[kg/m^3]$ and distance between the plates of $0.001[m]$.*

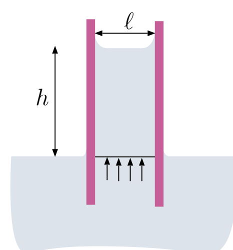



*Fig. -1.17. Capilary rise between two plates.*

Solution

In Figure 1.17 exhibits the liquid under the current study. The vertical forces acting on the body are the gravity, the pressure above and below and surface tension. It can be noted that the pressure and above are the same with the exception of the curveture on the upper part. Thus, the contol volume is taken just above the liquid and the air part is neglected. The question when the curveture should be ansered in the Dimentional analysis and for simplification this effect is neglected. The net forces in the vertical direction (positive upwords) per unit length are

$$2\sigma\cos 0^\circ = g\,h\,\ell\,\rho \Longrightarrow h = \frac{2\,\sigma}{\ell\,\rho\,g} \tag{1.51}$$

Inserting the values into equation (1.51) resutls in

$$h = \frac{2 \times 0.05}{0.001 \times 9.8 \times \times 1000} = \tag{1.52}$$

End Solution

Example 1.19:
*Develop expression for rise of the liquid due to surface tension in concentric cylinders.*

SOLUTION

The difference lie in the fact that "missing" cylinder add additional force and reduce the amount of liquid that has to raise. The balance between gravity and surface tension is

$$\sigma\, 2\,\pi\, \left(r_i\, \cos\theta_i + r_o \cos\theta_o\right) = \rho\, g\, h\, \left(\pi (r_o)^2 - \pi (r_i)^2\right) \tag{1.XIX.a}$$

Which can be simplified as

$$h = \frac{2\,\sigma\, (r_i\, \cos\theta_i + r_o \cos\theta_o)}{\rho\, g\, \left((r_o)^2 - (r_i)^2\right)} \tag{1.XIX.b}$$

The maximum is obtained when $\cos\theta_i = \cos\theta_o = 1$. Thus, equation (1.XIX.b) can be simplified

$$h = \frac{2\,\sigma}{\rho\, g\, (r_o - r_i)} \tag{1.XIX.c}$$

End Solution

### 1.7.1 Wetting of Surfaces

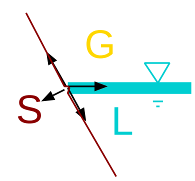



*Fig. -1.18. Forces in Contact angle.*

To explain the source of the contact angle, consider the point where three phases became in contact. This contact point occurs due to free surface reaching a solid boundary. The surface tension occurs between gas phase (G) to liquid phase (L) and also occurs between the solid (S) and the liquid phases as well as between the gas phase and the solid phase. In Figure 1.18, forces diagram is shown when control volume is chosen so that the masses of the solid, liquid, and gas can be ignored. Regardless to the magnitude of the surface tensions (except to zero) the forces cannot be balanced for the description of straight lines. For example, forces balanced along the line of solid boundary is

$$\sigma_{gs} - \sigma_{ls} - \sigma_{lg} \cos\beta = 0 \tag{1.53}$$

and in the tangent direction to the solid line the forces balance is

$$F_{solid} = \sigma_{lg} \sin\beta \tag{1.54}$$

substituting equation (1.54) into equation (1.53) yields

$$\sigma_{gs} - \sigma_{ls} = \frac{F_{solid}}{\tan\beta} \tag{1.55}$$

For $\beta = \pi/2 \Longrightarrow \tan\beta = \infty$. Thus, the solid reaction force must be zero. The gas solid surface tension is different from the liquid solid surface tension and hence violating equation (1.53).

The surface tension forces must be balanced, thus, a contact angle is created to balance it. The contact angle is determined by whether the surface tension between the gas solid (gs) is larger or smaller then the surface tension of liquid solid (ls) and the local geometry. It must be noted that the solid boundary isn't straight. The surface tension is a molecular phenomenon, thus depend on the locale structure of the surface and it provides the balance for these local structures.

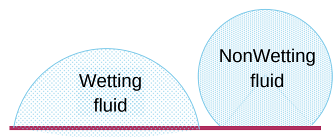



*Fig. -1.19. Description of wetting and non–wetting fluids.*

The connection of the three phases–materials–mediums creates two situations which are categorized as wetting or non–wetting. There is a common definition of wetting the surface. If the angle of the contact between three materials is larger than $90°$ then it is non-wetting. On the other hand, if the angle is below than $90°$ the material is wetting the surface (see Figure 1.19). The angle is determined by properties of the liquid, gas medium and the solid surface. And a small change on the solid surface can change the wetting condition to non–wetting. In fact there are commercial sprays that are intent to change the surface from wetting to non wetting. This fact is the reason that no reliable data can be provided with the exception to pure substances and perfect geometries. For example, water is described in many books as a wetting fluid. This statement is correct in most cases, however, when solid surface is made or cotted with certain materials, the water is changed to be wetting (for example 3M selling product to "change" water to non–wetting). So, the wetness of fluids is a function of the solid as well.

*Table -1.6. The contact angle for air, distilled water with selected materials to demonstrate the inconsistency.*

| chemical component | Contact Angle | Source |
|---|---|---|
| Steel | $\pi/3.7$ | [1] |
| Steel,Nickel | $\pi/4.74$ | [2] |
| Nickel | $\pi/4.74$ to $\pi/3.83$ | [1] |
| Nickel | $\pi/4.76$ to $\pi/3.83$ | [3] |
| Chrome-Nickel Steel | $\pi/3.7$ | [4] |
| Silver | $\pi/6$ to $\pi/4.5$ | [5] |

*Table -1.6. The contact angle for air, distilled water with selected materials to demonstrate the inconsistency. (continue)*

| chemical component | Contact Angle $\frac{mN}{m}$ | Source |
|---|---|---|
| Zink | $\pi/3.4$ | [4] |
| Bronze | $\pi/3.2$ | [4] |
| Copper | $\pi/4$ | [4] |
| Copper | $\pi/3$ | [7] |
| Copper | $\pi/2$ | [8] |

1 R. Siegel, E. G. Keshock (1975) "Effects of reduced gravity on nucleate boiling bubble dynamics in saturated water," AIChE Journal Volume 10 Issue 4, Pages 509 - 517. 1975

2 Bergles A. E. and Rohsenow W. M. "The determination of forced convection surface–boiling heat transfer, ASME, J. Heat Transfer, vol 1 pp 365 - 372.

3 Tolubinsky, V.I. and Ostrovsky, Y.N. (1966) "On the mechanism of boiling heat transfer",. International Journal of Heat and Mass Transfer, Vol. 9, No 12, pages 1465-1470.

4 Arefeva E.I., Aladev O, I.T., (1958) "wlijanii smatchivaemosti na teploobmen pri kipenii," Injenerno Fizitcheskij Jurnal, 11-17 1(7) In Russian.

5 Labuntsov D. A. (1939) "Approximate theory of heat transfer by developed nucleate boiling" In Sussian Izvestiya An SSSR , Energetika I transport, No 1.

6 Basu, N., Warrier, G. R., and Dhir, V. K., (2002) "Onset of Nucleate Boiling and Active Nucleation Site Density during Subcooled Flow Boiling," ASME Journal of Heat Transfer, Vol. 124, papes 717 -728.

7 Gaetner, R. F., and Westwater, J. W., (1960) "Population of Active Sites in Nucleate Boiling Heat Transfer," Chem. Eng. Prog. Symp., Ser. 56.

8 Wang, C. H., and Dhir, V. K., (1993), "Effect of Surface Wettability on Active Nucleation Site Density During Pool Boiling of Water on a Vertical Surface," J. Heat Transfer 115, pp. 659-669

To explain the contour of the surface, and the contact angle consider simple "wetting" liquid contacting a solid material in two–dimensional shape as depicted in Figure 1.20. To solve the shape of the liquid surface, the pressure difference between the two sides of free surface has to be balanced by the surface tension. In Figure 1.20 describes the raising

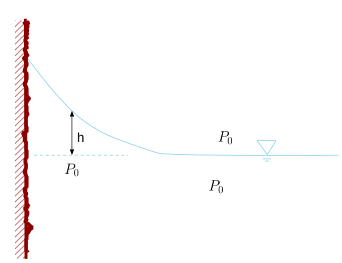



*Fig. -1.20. Description of liquid surface.*

of the liquid as results of the surface tension. The surface tension reduces the pressure in the liquid above the liquid line (the dotted line in the Figure 1.20). The pressure just below the surface is $-g\,h(x)\,\rho$ (this pressure difference will be explained in more details in Chapter 4). The pressure, on the gas side, is the atmospheric pressure. This problem is a two dimensional problem and equation (1.46) is applicable to it. Appalling equation (1.46) and using the pressure difference yields

$$g\,h(x), \rho = \frac{\sigma}{R(x)} \tag{1.56}$$

The radius of any continuous function, h = h(x), is

$$R(x) = \frac{\left(1 + \left[\dot{h}(x)\right]^2\right)^{3/2}}{\ddot{h}(x)} \tag{1.57}$$

Where $\dot{h}$ is the derivative of $h$ with respect to $x$.

Equation (1.57) can be derived either by forcing a circle at three points at (x, x+dx, and x+2dx) and thus finding the the diameter or by geometrical analysis of triangles build on points x and x+dx (perpendicular to the tangent at these points). Substituting equation (1.57) into equation (1.56) yields

$$g\,h(x)\,\rho = \frac{\sigma}{\dfrac{\left(1 + \left[\dot{h}(x)\right]^2\right)^{3/2}}{\ddot{h}(x)}} \tag{1.58}$$

Equation (1.58) is non–linear differential equation for height and can be written as

**1-D Surface Due to Surface Tension**

$$\frac{g\,h\,\rho}{\sigma}\left(1 + \left[\frac{dh}{dx}\right]^2\right)^{3/2} - \frac{d^2h}{dx^2} = 0 \tag{1.59}$$

With the boundary conditions that specify either the derivative $\dot{h}(x = r) = 0$ (symmetry) and the derivative at $\dot{h}x = \beta$ or heights in two points or other combinations. An alternative presentation of equation (1.58) is

$$g\,h\,\rho = \frac{\sigma\ddot{h}}{\left(1 + \dot{h}^2\right)^{3/2}} \tag{1.60}$$

Integrating equation (1.60) transforms into

$$\int \frac{g\,\rho}{\sigma} h\,dh = \int \frac{\ddot{h}}{\left(1+\dot{h}^2\right)^{3/2}} dh \tag{1.61}$$

The constant $Lp\,\sigma/\rho\,g$ is referred to as Laplace's capillarity constant. The units of this constant are meter squared. The differential $dh$ is $\dot{h}$. Using dummy variable and the identities $\dot{h} = \xi$ and hence, $\ddot{h} = \dot{\xi} = d\xi$ transforms equation (1.61) into

$$\int \frac{1}{Lp} h\,dh = \int \frac{\xi d\xi}{\left(1+\xi^2\right)^{3/2}} \tag{1.62}$$

After the integration equation (1.62) becomes

$$\frac{h^2}{2\,Lp} + constant = -\frac{1}{\left(1+\dot{h}^2\right)^{1/2}} \tag{1.63}$$

At infinity, the height and the derivative of the height must by zero so $constant + 0 = -1/1$ and hence, $constant = -1$ .

$$1 - \frac{h^2}{2\,Lp} = \frac{1}{\left(1+\dot{h}^2\right)^{1/2}} \tag{1.64}$$

Equation (1.64) is a first order differential equation that can be solved by variables separation[5]. Equation (1.64) can be rearranged to be

$$\left(1+\dot{h}^2\right)^{1/2} = \frac{1}{1-\frac{h^2}{2\,Lp}} \tag{1.65}$$

Squaring both sides and moving the one to the right side yields

$$\dot{h}^2 = \left(\frac{1}{1-\frac{h^2}{2\,Lp}}\right)^2 - 1 \tag{1.66}$$

The last stage of the separation is taking the square root of both sides to be

$$\dot{h} = \frac{dh}{dx} = \sqrt{\left(\frac{1}{1-\frac{h^2}{2\,Lp}}\right)^2 - 1} \tag{1.67}$$

[5]This equation has an analytical solution which is $x = Lp\sqrt{4-(h/Lp)^2} - Lp\,acosh(2\,Lp/h) + constant$ where $Lp$ is the Laplace constant. Shamefully, this author doesn't know how to show it in a two lines derivations.

or

$$\frac{dh}{\sqrt{\left(\frac{1}{1-\frac{h^2}{2\,Lp}}\right)^2 - 1}} = dx \tag{1.68}$$

Equation (1.68) can be integrated to yield

$$\int \frac{dh}{\sqrt{\left(\frac{1}{1-\frac{h^2}{2\,Lp}}\right)^2 - 1}} = \text{x} + \text{constant} \tag{1.69}$$

The constant is determined by the boundary condition at $x = 0$. For example if $h(x-0) = h_0$ then $constant = h_0$. This equation is studied extensively in classes on surface tension. Furthermore, this equation describes the dimensionless parameter that affects this phenomenon and this parameter will be studied in Chapter ?. This book is introductory, therefore this discussion on surface tension equation will be limited.

#### 1.7.1.1 Capillarity

The capillary forces referred to the fact that surface tension causes liquid to rise or penetrate into area (volume), otherwise it will not be there. It can be shown that the height that the liquid raised in a tube due to the surface tension is

$$h = \frac{2\,\sigma\cos\beta}{g\,\Delta\rho\,r} \tag{1.70}$$

Where $\Delta\rho$ is the difference of liquid density to the gas density and $r$ is the radius of tube.

But this simplistic equation is unusable and useless unless the contact angle (assuming that the contact angel is constant or a repressive average can be found or provided or can be measured) is given. However, in reality there is no readily information for contact angle[6] and therefore this equation is useful to show the treads. The maximum that the contact angle can be obtained in equation (1.70) when $\beta = 0$ and thus $\cos\beta = 1$. This angle is obtained when a perfect half a sphere shape exist of the liquid surface. In that case equation (1.70) becomes

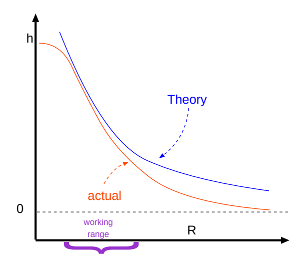



*Fig. -1.21. The raising height as a function of the radii.*

$$h_{max} = \frac{2\,\sigma}{g\,\Delta\rho\,r} \tag{1.71}$$

[6]Actually, there are information about the contact angle. However, that information conflict each other and no real information is available see Table 1.6.

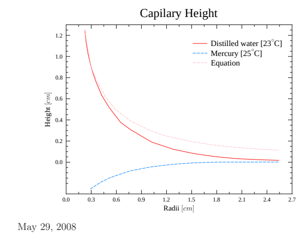



*Fig. -1.22. The raising height as a function of the radius.*

Figure 1.22 exhibits the height as a function of the radius of the tube. The height based on equation (1.71) is shown in Figure 1.21 as blue line. The actual height is shown in the red line. Equation (1.71) provides reasonable results only in a certain range. For a small tube radius, equation (1.59) proved better results because the curve approaches hemispherical sphere (small gravity effect). For large radii equation (1.59) approaches the strait line (the liquid line) strong gravity effect. On the other hand, for extremely small radii equation (1.71) indicates that the high height which indicates a negative pressure. The liquid at a certain pressure will be vaporized and will breakdown the model upon this equation was constructed. Furthermore, the small scale indicates that the simplistic and continuous approach is not appropriate and a different model is needed. The conclusion of this discussion are shown in Figure 1.21. The actual dimension for many liquids (even water) is about 1-5 $[mm]$.

The discussion above was referred to "wetting" contact angle. The depression of the liquid occurs in a "negative" contact angle similarly to "wetting." The depression height, $h$ is similar to equation (1.71) with a minus sign. However, the gravity is working against the surface tension and reducing the range and quality of the predictions of equation (1.71). The measurements of the height of distilled water and mercury are presented in Figure 1.22. The experimental results of these materials are with agreement with the discussion above.

The surface tension of a selected material is given in Table 1.7.

In conclusion, the surface tension issue is important only in case where the radius is very small and gravity is negligible. The surface tension depends on the two materials or mediums that it separates.

Example 1.20:

*Calculate the diameter of a water droplet to attain pressure difference of 1000$[N/m^2]$. You can assume that temperature is $20°C$.*

Solution

The pressure inside the droplet is given by equation (1.47).

$$D = 2\,R = \frac{2\,2\,\sigma}{\Delta P} = \frac{4 \times 0.0728}{1000} \sim 2.912\,10^{-4}[m]$$

End Solution

Example 1.21:

*Calculate the pressure difference between a droplet of water at $20°C$ when the droplet has a diameter of 0.02 cm.*

SOLUTION

using equation

$$\Delta P = \frac{2\,\sigma}{r} \sim \frac{2 \times 0.0728}{0.0002} \sim 728.0[N/m^2]$$

End Solution

Example 1.22:

*Calculate the maximum force necessary to lift a thin wire ring of 0.04[m] diameter from a water surface at $20°C$. Neglect the weight of the ring.*

SOLUTION

$$F = 2(2\,\pi\,r\,\sigma)\cos\beta$$

The actual force is unknown since the contact angle is unknown. However, the maximum Force is obtained when $\beta = 0$ and thus $\cos\beta = 1$. Therefore,

$$F = 4\,\pi\,r\,\sigma = 4 \times \pi \times 0.04 \times 0.0728 \sim .0366[N]$$

In this value the gravity is not accounted for.

End Solution

Example 1.23:

*A small liquid drop is surrounded with the air and has a diameter of 0.001 [m]. the pressure difference between the inside and outside droplet is 1[kPa]. Estimate the surface tension?*

SOLUTION

To be continue

End Solution

*Table -1.7. The surface tension for selected materials at temperature $20°C$ when not mentioned.*

| chemical component | Surface Tension $\frac{mN}{m}$ | $T$ | correction $\frac{mN}{m\,K}$ |
|---|---|---|---|
| Acetic Acid | 27.6 | $20°C$ | n/a |
| Acetone | 25.20 | - | -0.1120 |
| Aniline | 43.4 | $22°C$ | -0.1085 |
| Benzene | 28.88 | - | -0.1291 |
| Benzylalcohol | 39.00 | - | -0.0920 |
| Benzylbenzoate | 45.95 | - | -0.1066 |
| Bromobenzene | 36.50 | - | -0.1160 |

*Table -1.7. The surface tension for selected materials (continue)*

| chemical component | Surface Tension $\frac{mN}{m}$ | $T$ | correction $\frac{mN}{m\,K}$ |
|---|---|---|---|
| Bromobenzene | 36.50 | - | -0.1160 |
| Bromoform | 41.50 | - | -0.1308 |
| Butyronitrile | 28.10 | - | -0.1037 |
| Carbon disulfid | 32.30 | - | -0.1484 |
| Quinoline | 43.12 | - | -0.1063 |
| Chloro benzene | 33.60 | - | -0.1191 |
| Chloroform | 27.50 | - | -0.1295 |
| Cyclohexane | 24.95 | - | -0.1211 |
| Cyclohexanol | 34.40 | $25°C$ | -0.0966 |
| Cyclopentanol | 32.70 | - | -0.1011 |
| Carbon Tetrachloride | 26.8 | - | n/a |
| Carbon disulfid | 32.30 | - | -0.1484 |
| Chlorobutane | 23.10 | - | -0.1117 |
| Ethyl Alcohol | 22.3 | - | n/a |
| Ethanol | 22.10 | - | -0.0832 |
| Ethylbenzene | 29.20 | - | -0.1094 |
| Ethylbromide | 24.20 | - | -0.1159 |
| Ethylene glycol | 47.70 | - | -0.0890 |
| Formamide | 58.20 | - | -0.0842 |
| Gasoline | $\sim$ 21 | - | n/a |
| Glycerol | 64.0 | - | -0.0598 |
| Helium | 0.12 | $-269°C$ | n/a |
| Mercury | 425-465.0 | - | -0.2049 |
| Methanol | 22.70 | - | -0.0773 |
| Methyl naphthalene | 38.60 | - | -0.1118 |
| Methyl Alcohol | 22.6 | - | n/a |
| Neon | 5.15 | $-247°C$ | n/a |
| Nitrobenzene | 43.90 | - | -0.1177 |
| Olive Oil | 43.0-48.0 | - | -0.067 |
| Perfluoroheptane | 12.85 | - | -0.0972 |
| Perfluorohexane | 11.91 | - | -0.0935 |
| Perfluorooctane | 14.00 | - | -0.0902 |
| Phenylisothiocyanate | 41.50 | - | -0.1172 |
| Propanol | 23.70 | $25°C$ | -0.0777 |
| Pyridine | 38.00 | - | -0.1372 |
| Pyrrol | 36.60 | - | -0.1100 |
| SAE 30 Oil | n/a | - | n/a |
| Seawater | 54-69 | - | n/a |
| Toluene | 28.4 | - | -0.1189 |

*Table -1.7. The surface tension for selected materials (continue)*

| chemical component | Surface Tension $\frac{mN}{m}$ | $T$ | correction $\frac{mN}{m\,K}$ |
|---|---|---|---|
| Turpentine | 27 | - | n/a |
| Water | 72.80 | - | -0.1514 |
| o-Xylene | 30.10 | - | -0.1101 |
| m-Xylene | 28.90 | - | -0.1104 |

# CHAPTER 2

# Review of Thermodynamics

In this chapter, a review of several definitions of common thermodynamics terms is presented. This introduction is provided to bring the student back to current place with the material.

## 2.1 Basic Definitions

The following basic definitions are common to thermodynamics and will be used in this book.

**Work**

In mechanics, the work was defined as

$$\text{mechanical work} = \int \mathbf{F} \bullet \mathbf{d}\ell = \int P dV \tag{2.1}$$

This definition can be expanded to include two issues. The first issue that must be addressed, that work done on the surroundings by the system boundaries similarly is positive. Two, there is a transfer of energy so that its effect can cause work. It must be noted that electrical current is a work while heat transfer isn't.

**System**

This term will be used in this book and it is defined as a continuous (at least partially) fixed quantity of matter. The dimensions of this material can be changed. In this definition, it is assumed that the system speed is significantly lower than that of the speed of light. So, the mass can be assumed constant even though the true conservation law applied to the combination of mass energy (see Einstein's law). In fact for almost all engineering purpose this law is reduced to two separate laws of mass conservation and energy conservation.

Our system can receive energy, work, etc as long the mass remain constant the definition is not broken.

**Thermodynamics First Law**

This law refers to conservation of energy in a non accelerating system. Since all the systems can be calculated in a non accelerating systems, the conservation is applied to all systems. The statement describing the law is the following.

$$Q_{12} - W_{12} = E_2 - E_1 \tag{2.2}$$

The system energy is a state property. From the first law it directly implies that for process without heat transfer (adiabatic process) the following is true

$$W_{12} = E_1 - E_2 \tag{2.3}$$

Interesting results of equation (2.3) is that the way the work is done and/or intermediate states are irrelevant to final results. There are several definitions/separations of the kind of works and they include kinetic energy, potential energy (gravity), chemical potential, and electrical energy, etc. The internal energy is the energy that depends on the other properties of the system. For example for pure/homogeneous and simple gases it depends on two properties like temperature and pressure. The internal energy is denoted in this book as $E_U$ and it will be treated as a state property.

The potential energy of the system is depended on the body force. A common body force is the gravity. For such body force, the potential energy is $mgz$ where $g$ is the gravity force (acceleration), $m$ is the mass and the $z$ is the vertical height from a datum. The kinetic energy is

$$K.E. = \frac{mU^2}{2} \tag{2.4}$$

Thus the energy equation can be written as

Total Energy Equation

$$\frac{m{U_1}^2}{2} + mgz_1 + E_{U\,1} + Q = \frac{m{U_2}^2}{2} + mgz_2 + E_{U\,2} + W \tag{2.5}$$

For the unit mass of the system equation (2.5) is transformed into

Spesific Energy Equation

$$\frac{{U_1}^2}{2} + gz_1 + E_{u\,1} + q = \frac{{U_2}^2}{2} + gz_2 + E_{u\,2} + w \tag{2.6}$$

where $q$ is the energy per unit mass and $w$ is the work per unit mass. The "new" internal energy, $E_u$, is the internal energy per unit mass.

Since the above equations are true between arbitrary points, choosing any point in time will make it correct. Thus differentiating the energy equation with respect to time yields the rate of change energy equation. The rate of change of the energy transfer is

$$\frac{DQ}{Dt} = \dot{Q} \tag{2.7}$$

In the same manner, the work change rate transfered through the boundaries of the system is

$$\frac{DW}{Dt} = \dot{W} \tag{2.8}$$

Since the system is with a fixed mass, the rate energy equation is

$$\dot{Q} - \dot{W} = \frac{D\,E_U}{Dt} + mU\frac{DU}{Dt} + m\frac{D\,B_f\,z}{Dt} \tag{2.9}$$

For the case were the body force, $B_f$, is constant with time like in the case of gravity equation (2.9) reduced to

Time Dependent Energy Equation

$$\dot{Q} - \dot{W} = \frac{D\,E_U}{Dt} + m\,U\frac{DU}{Dt} + m\,g\frac{D\,z}{Dt} \tag{2.10}$$

The time derivative operator, $D/Dt$ is used instead of the common notation because it referred to system property derivative.

**Thermodynamics Second Law**

There are several definitions of the second law. No matter which definition is used to describe the second law it will end in a mathematical form. The most common mathematical form is Clausius inequality which state that

$$\oint \frac{\delta Q}{T} \geq 0 \tag{2.11}$$

The integration symbol with the circle represent integral of cycle (therefor circle) in with system return to the same condition. If there is no lost, it is referred as a reversible process and the inequality change to equality.

$$\oint \frac{\delta Q}{T} = 0 \tag{2.12}$$

The last integral can go though several states. These states are independent of the path the system goes through. Hence, the integral is independent of the path. This observation leads to the definition of entropy and designated as $S$ and the derivative of entropy is

$$ds \equiv \left(\frac{\delta Q}{T}\right)_{\text{rev}} \tag{2.13}$$

Performing integration between two states results in

$$S_2 - S_1 = \int_1^2 \left(\frac{\delta Q}{T}\right)_{\text{rev}} = \int_1^2 dS \tag{2.14}$$

One of the conclusions that can be drawn from this analysis is for reversible and adiabatic process $dS = 0$. Thus, the process in which it is reversible and adiabatic, the entropy remains constant and referred to as isentropic process. It can be noted that there is a possibility that a process can be irreversible and the right amount of heat transfer to have zero change entropy change. Thus, the reverse conclusion that zero change of entropy leads to reversible process, isn't correct.

For reversible process equation (2.12) can be written as

$$\delta Q = T dS \tag{2.15}$$

and the work that the system is doing on the surroundings is

$$\delta W = P dV \tag{2.16}$$

Substituting equations (2.15) (2.16) into (2.10) results in

$$T dS = d\,E_U + P dV \tag{2.17}$$

Even though the derivation of the above equations were done assuming that there is no change of kinetic or potential energy, it still remail valid for all situations. Furthermore, it can be shown that it is valid for reversible and irreversible processes.

**Enthalpy**

It is a common practice to define a new property, which is the combination of already defined properties, the enthalpy of the system.

$$H = E_U + PV \tag{2.18}$$

The specific enthalpy is enthalpy per unit mass and denoted as, $h$.

Or in a differential form as

$$dH = dE_U + dP\,V + P\,dV \tag{2.19}$$

Combining equations (2.18) the (2.17) yields

(one form of) Gibbs Equation

$$T dS = dH - V dP \tag{2.20}$$

For isentropic process, equation (2.17) is reduced to $dH = V dP$. The equation (2.17) in mass unit is

$$T ds = du + P dv = dh - \frac{dP}{\rho} \tag{2.21}$$

when the density enters through the relationship of $\rho = 1/v$.

**Specific Heats**

The change of internal energy and enthalpy requires new definitions. The first change of the internal energy and it is defined as the following

Spesific Volume Heat

$$C_v \equiv \left(\frac{\partial E_u}{\partial T}\right) \tag{2.22}$$

And since the change of the enthalpy involve some kind of work is defined as

Spesific Pressure Heat

$$C_p \equiv \left(\frac{\partial h}{\partial T}\right) \tag{2.23}$$

The ratio between the specific pressure heat and the specific volume heat is called the ratio of the specific heat and it is denoted as, $k$.

Spesific Heats Ratio

$$k \equiv \frac{C_p}{C_v} \tag{2.24}$$

For solid, the ratio of the specific heats is almost 1 and therefore the difference between them is almost zero. Commonly the difference for solid is ignored and both are assumed to be the same and therefore referred as $C$. This approximation less strong for liquid but not by that much and in most cases it applied to the calculations. The ratio the specific heat of gases is larger than one.

**Equation of state**

Equation of state is a relation between state variables. Normally the relationship of temperature, pressure, and specific volume define the equation of state for gases. The simplest equation of state referred to as ideal gas. and it is defined as

$$P = \rho R T \tag{2.25}$$

Application of Avogadro's law, that "all gases at the same pressures and temperatures have the same number of molecules per unit of volume," allows the calculation of a "universal gas constant." This constant to match the standard units results in

$$\bar{R} = 8.3145 \frac{kj}{kmol\ K} \tag{2.26}$$

Thus, the specific gas can be calculate as

$$R = \frac{\bar{R}}{M} \tag{2.27}$$

The specific constants for select gas at 300K is provided in table 2.1.

*Table -2.1. Properties of Various Ideal Gases [300K]*

| Gas | Chemical Formula | Molecular Weight | R $\left[\frac{kj}{KgK}\right]$ | $C_P \left[\frac{kj}{KgK}\right]$ | $C_v \left[\frac{kj}{KgK}\right]$ | $k$ |
|---|---|---|---|---|---|---|
| Air | - | 28.970 | 0.28700 | 1.0035 | 0.7165 | 1.400 |
| Argon | Ar | 39.948 | 0.20813 | 0.5203 | 0.3122 | 1.667 |
| Butane | $C_4H_{10}$ | 58.124 | 0.14304 | 1.7164 | 1.5734 | 1.091 |
| Carbon Dioxide | $CO_2$ | 44.01 | 0.18892 | 0.8418 | 0.6529 | 1.289 |
| Carbon Monoxide | $CO$ | 28.01 | 0.29683 | 1.0413 | 0.7445 | 1.400 |
| Ethane | $C_2H_6$ | 30.07 | 0.27650 | 1.7662 | 1.4897 | 1.186 |
| Ethylene | $C_2H_4$ | 28.054 | 0.29637 | 1.5482 | 1.2518 | 1.237 |
| Helium | $He$ | 4.003 | 2.07703 | 5.1926 | 3.1156 | 1.667 |
| Hydrogen | $H_2$ | 2.016 | 4.12418 | 14.2091 | 10.0849 | 1.409 |
| Methane | $CH_4$ | 16.04 | 0.51835 | 2.2537 | 1.7354 | 1.299 |
| Neon | $Ne$ | 20.183 | 0.41195 | 1.0299 | 0.6179 | 1.667 |
| Nitrogen | $N_2$ | 28.013 | 0.29680 | 1.0416 | 0.7448 | 1.400 |
| Octane | $C_8H_{18}$ | 114.230 | 0.07279 | 1.7113 | 1.6385 | 1.044 |
| Oxygen | $O_2$ | 31.999 | 0.25983 | 0.9216 | 0.6618 | 1.393 |
| Propane | $C_3H_8$ | 44.097 | 0.18855 | 1.6794 | 1.4909 | 1.126 |
| Steam | $H_2O$ | 18.015 | 0.48152 | 1.8723 | 1.4108 | 1.327 |

From equation (2.25) of state for perfect gas it follows

$$d(Pv) = RdT \tag{2.28}$$

For perfect gas

$$dh = dE_u + d(Pv) = dE_u + d(RT) = f(T) \text{ (only)} \tag{2.29}$$

From the definition of enthalpy it follows that

$$d(Pv) = dh - dE_u \tag{2.30}$$

Utilizing equation (2.28) and subsisting into equation (2.30) and dividing by $dT$ yields

$$C_p - C_v = R \tag{2.31}$$

This relationship is valid only for ideal/perfect gases.

The ratio of the specific heats can be expressed in several forms as

**$C_v$ to Spesific Heats Ratio**

$$C_v = \frac{R}{k-1} \tag{2.32}$$

**$C_p$ to Spesific Heats Ratio**

$$C_p = \frac{k\,R}{k-1} \tag{2.33}$$

The specific heat ratio, $k$ value ranges from unity to about 1.667. These values depend on the molecular degrees of freedom (more explanation can be obtained in Van Wylen "F. of Classical thermodynamics." The values of several gases can be approximated as ideal gas and are provided in Table (2.1).

The entropy for ideal gas can be simplified as the following

$$s_2 - s_1 = \int_1^2 \left( \frac{dh}{T} - \frac{dP}{\rho T} \right) \tag{2.34}$$

Using the identities developed so far one can find that

$$s_2 - s_1 = \int_1^2 C_p \frac{dT}{T} - \int_1^2 \frac{R\,dP}{P} = C_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1} \tag{2.35}$$

Or using specific heat ratio equation (2.35) transformed into

$$\frac{s_2 - s_1}{R} = \frac{k}{k-1} \ln \frac{T_2}{T_1} - \ln \frac{P_2}{P_1} \tag{2.36}$$

For isentropic process, $\Delta s = 0$, the following is obtained

$$\ln \frac{T_2}{T_1} = \ln \left( \frac{P_2}{P_1} \right)^{\frac{k-1}{k}} \tag{2.37}$$

There are several famous identities that results from equation (2.37) as

**Ideal Gas Isontropic Relationships**

$$\frac{T_2}{T_1} = \left( \frac{P_2}{P_1} \right)^{\frac{k-1}{k}} = \left( \frac{V_1}{V_2} \right)^{k-1} \tag{2.38}$$

The ideal gas model is a simplified version of the real behavior of real gas. The real gas has a correction factor to account for the deviations from the ideal gas model. This correction factor referred as the compressibility factor and defined as

Z deviation from the Ideal Gas Model

$$Z = \frac{PV}{RT} \tag{2.39}$$

# CHAPTER 3

# Review of Mechanics

*This author would like to express his gratitude to Dan Olsen (former Minneapolis city Engineer) and his friend Richard Hackbarth.*

This chapter provides a review of important definitions and concepts from Mechanics (statics and dynamics). These concepts and definitions will be used in this book and a review is needed.

## 3.1 Kinematics of of Point Body

A point body is location at time, $t$ in a location, $\vec{\boldsymbol{R}}$. The velocity is derivative of the change of the location and using the chain role (for the direction and one for the magnitude) results,

$$\vec{\boldsymbol{U}} = \frac{d\vec{\boldsymbol{R}}}{dt} = \overbrace{\left.\frac{d\vec{\boldsymbol{R}}}{dt}\right|_R}^{\text{change in } R \text{ direction}} + \overbrace{\vec{\omega} \times \vec{\boldsymbol{R}}}^{\text{change in perpendicular to R}} \tag{3.1}$$

Notice that $\vec{\omega}$ can have three dimensional components. It also can be noticed that this derivative is present derivation of any victory. The acceleration is the derivative of the velocity

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{U}}}{dt} = \overbrace{\left.\frac{d^2\vec{\boldsymbol{R}}}{dt^2}\right|_R}^{\text{“regular acceleration”}} + \overbrace{\left(\vec{\boldsymbol{R}} \times \frac{d\vec{\omega}}{dt}\right)}^{\text{angular acceleration}} + \overbrace{\vec{\omega} \times \left(\vec{\boldsymbol{R}} \times \vec{\omega}\right)}^{\text{centrifugal acceleration}} + \overbrace{2\left(\left.\frac{d\vec{\boldsymbol{R}}}{dt}\right|_R \times \omega\right)}^{\text{Coriolis acceleration}} \tag{3.2}$$

Example 3.1:

*A water jet is supposed be used to extinguish the fire in a building as depicted in Figure*

*3.1[1]. For given velocity, at what angle the jet has to be shot so that velocity will be horizontal at the window. Assume that gravity is $g$ and the distance of the nozzle from the building is $a$ and height of the window from the nozzle is $b$. To simplify the calculations, it proposed to calculate the velocity of the point particle to toward the window. Calculate what is the velocity so that the jet reach the window. What is the angle that jet has to be aimed.*

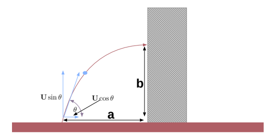



*Fig. -3.1. Description of the extinguish nozzle aimed at the building window.*

SOLUTION

The initial velocity is unknown and denoted as $U$ which two components. The velocity at $x$ is $U_x = U\cos\theta$ and the velocity in $y$ direction is $U_y = U\sin\theta$. There there are three unknowns, $U$, $\theta$, and time, $t$ and three equations. The equation for the $x$ coordinate is

$$a = U\cos\theta\, t \tag{3.I.a}$$

The distance for $y$ equation for coordinate (zero is at the window) is

$$0 = -\frac{g\,t^2}{2} + U\,\sin\theta\, t - b \tag{3.I.b}$$

The velocity for the $y$ coordinate at the window is zero

$$u(t) = 0 = -g\,t + U\,\sin\theta \tag{3.I.c}$$

These nonlinear equations (3.I.a), (3.I.b) and (3.I.c) can be solved explicitly. Isolating $t$ from (3.I.a) and substituting into equations (3.I.b) and (3.I.c)

$$b = \frac{-g\,a^2}{2\,U^2\,cos^2\theta} + a\,\tan\theta \tag{3.I.d}$$

and equation (3.I.a) becomes

$$0 = \frac{-g\,a}{U\,\cos\theta} + U\,\cos\theta \Longrightarrow U = \frac{\sqrt{a\,g}}{\cos\theta} \tag{3.I.e}$$

Substituting (3.I.e) into (3.I.d) results in

$$\tan\theta = \frac{b}{a} + \frac{1}{2} \tag{3.I.f}$$

End Solution

[1]While the simple example does not provide exact use of the above equation it provides experience of going over the motions of kinematics.

## 3.2 Center of Mass

The center of mass is divided into two sections, first, center of the mass and two, center of area (two–dimensional body with equal distribution mass).

### 3.2.1 Actual Center of Mass

In many engineering problems, the center of mass is required to make the calculations. This concept is derived from the fact that a body has a center of mass/gravity which interacts with other bodies and that this force acts on the center (equivalent force). It turns out that this concept is very useful in calculating rotations, moment of inertia, etc. The center of mass doesn't depend on the coordinate system and on the way it is calculated. The physical meaning of the center of mass is that if a straight line force acts on the body in away through the center of gravity, the body will not rotate. In other words, if a body will be held by one point it will be enough to hold the body in the direction of the center of mass. Note, if the body isn't be held through the center of mass, then a moment in additional to force is required (to prevent the body for rotating). It is convenient to use the Cartesian system to explain this concept. Suppose that the body has a distribution of the mass (density, $rho$) as a function of the location. The density "normally" defined as mass per volume. Here, the the line density is referred to density mass per unit length in the $x$ direction.

In $x$ coordinate, the center will be defined as

$$\bar{x} = \frac{1}{m} \int_V x \, \overbrace{\rho(x) dV}^{dm} \tag{3.3}$$

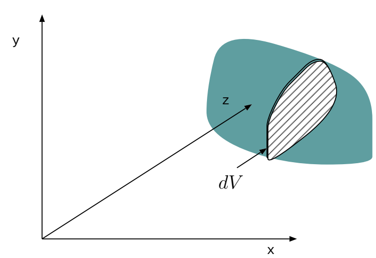



*Fig. -3.2. Description of how the center of mass is calculated.*

Here, the $dV$ element has finite dimensions in y–z plane and infinitesimal dimension in $x$ direction see Figure 3.2. Also, the mass, $m$ is the total mass of the object. It can be noticed that center of mass in the x–direction isn't affected by the distribution in the $y$ nor by $z$ directions. In same fashion the center of mass can be defined in the other directions as following

$x_i$ of Center Mass

$$\bar{x_i} = \frac{1}{m} \int_V x_i \rho(x_i) dV \tag{3.4}$$

where $x_i$ is the direction of either, $x$, $y$ or $z$. The density, $\rho(x_i)$ is the line density as function of $x_i$. Thus, even for solid and uniform density the line density is a function of the geometry.

### 3.2.2 Aproximate Center of Area

In the previous case, the body was a three dimensional shape. There are cases where the body can be approximated as a two-dimensional shape because the body is with a thin with uniform density. Consider a uniform thin body with constant thickness shown in Figure 3.3 which has density, $\rho$. Thus, equation (3.3) can be transferred into

$$\bar{x} = \frac{1}{\underbrace{t\,A}_{V}\,\rho} \int_V x \,\overbrace{\rho\,t\,dA}^{dm} \tag{3.5}$$

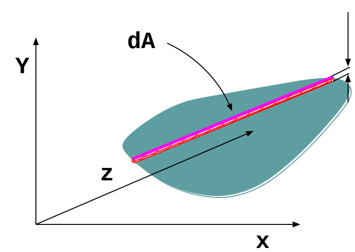



*Fig. -3.3. Thin body center of mass/area schematic.*

The density, $\rho$ and the thickness, $t$, are constant and can be canceled. Thus equation (3.5) can be transferred into

Aproxiate $x_i$ of Center Mass

$$\bar{x_i} = \frac{1}{A} \int_A x_i dA \tag{3.6}$$

when the integral now over only the area as oppose over the volume.

Finding the centroid location should be done in the most convenient coordinate system since the location is coordinate independent.

## 3.3 Moment of Inertia

As it was divided for the body center of mass, the moment of inertia is divided into moment of inertia of mass and area.

### 3.3.1 Moment of Inertia for Mass

The moment of inertia turns out to be an essential part for the calculations of rotating bodies. Furthermore, it turns out that the moment of inertia has much wider applicability. Moment of inertia of mass is defined as

Moment of Inertia

$$I_{rrm} = \int_m \rho r^2 dm \tag{3.7}$$

If the density is constant then equation (3.7) can be transformed into

$$I_{rrm} = \rho \int_V r^2 dV \tag{3.8}$$

The moment of inertia is independent of the coordinate system used for the calculation, but dependent on the location of axis of rotation relative to the body. Some people define the radius of gyration as an equivalent concepts for the center of mass concept and which means if all the mass were to locate in the one point/distance and to obtain the same of moment of inertia.

$$r_k = \sqrt{\frac{I_m}{m}} \tag{3.9}$$

The body has a different moment of inertia for every coordinate/axis and they are

$$\begin{aligned} I_{xx} &= \int_V {r_x}^2 dm = \int_V (y^2 + z^2)\, dm \\ I_{yy} &= \int_V {r_y}^2 dm = \int_V (x^2 + z^2)\, dm \\ I_{zz} &= \int_V {r_z}^2 dm = \int_V (x^2 + y^2)\, dm \end{aligned} \tag{3.10}$$

## 3.3.2 Moment of Inertia for Area

### 3.3.2.1 General Discussion

For body with thickness, $t$ and uniform density the following can be written

$$I_{xxm} = \int_m r^2 dm = \rho\, t \overbrace{\int_A r^2 dA}^{\text{moment of inertia for area}} \tag{3.11}$$

The moment of inertia about axis is $x$ can be defined as

Moment of Inertia

$$\boxed{I_{xx} = \int_A r^2 dA = \frac{I_{xxm}}{\rho\, t}} \tag{3.12}$$

where $r$ is distance of $dA$ from the axis $x$ and $t$ is the thickness. Any point distance can be calculated from axis $x$ as

$$x = \sqrt{y^2 + z^2} \tag{3.13}$$

Thus, equation (3.12) can be written as

$$I_{xx} = \int_A \left(y^2 + z^2\right) dA \tag{3.14}$$

In the same fashion for other two coordinates as

$$I_{yy} = \int_A \left(x^2 + z^2\right) dA \tag{3.15}$$

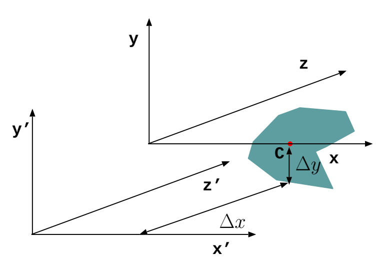



*Fig. -3.4. The schematic that explains the summation of moment of inertia.*

$$I_{zz} = \int_A \left(x^2 + y^2\right) dA \tag{3.16}$$

#### 3.3.2.2 The Parallel Axis Theorem

The moment of inertial can be calculated for any axis. The knowledge about one axis can help calculating the moment of inertia for a parallel axis. Let $I_{xx}$ the moment of inertia about axis $xx$ which is at the center of mass/area.

The moment of inertia for axis $x^{'}$ is

$$I_{x^{'}x^{'}} = \int_A {r^{'}}^2 dA = \int_A \left({y^{'}}^2 + {z^{'}}^2\right) dA = \int_A \left[(y+\Delta y)^2 + (z+\Delta z)^2\right] dA \tag{3.17}$$

equation (3.17) can be expended as

$$I_{x^{'}x^{'}} = \overbrace{\int_A \left(y^2 + z^2\right) dA}^{I_{xx}} + \overbrace{2\int_A \left(y\,\Delta y + z\,\Delta z\right) dA}^{=0} + \int_A \left((\Delta y)^2 + (\Delta z)^2\right) dA \tag{3.18}$$

The first term in equation (3.18) on the right hand side is the moment of inertia about axis $x$ and the second them is zero. The second therm is zero because it integral of center about center thus is zero. The third term is a new term and can be written as

$$\int_A \overbrace{\left((\Delta y)^2 + (\Delta z)^2\right)}^{constant} dA = \overbrace{\left((\Delta y)^2 + (\Delta z)\right)}^{r^2} \overbrace{\int_A^2 dA}^{A} = r^2\, A \tag{3.19}$$

Hence, the relationship between the moment of inertia at $xx$ and parallel axis $x^{'}x^{'}$ is

**Parallel Axis Equation**

$$I_{x^{'}x^{'}} = I_{xx} + r^2\, A \tag{3.20}$$

The moment of inertia of several areas is the sum of moment inertia of each area see Figure 3.5 and therefore,

$$I_{xx} = \sum_{i=1}^{n} I_{xx\,i} \tag{3.21}$$

If the same areas are similar thus

$$I_{xx} = \sum_{i=1}^{n} I_{xx\,i} = n\, I_{xx\,i} \tag{3.22}$$

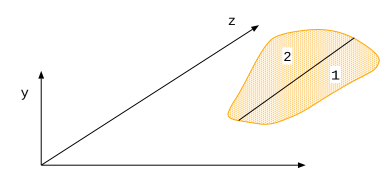



*Fig. -3.5. The schematic to explain the summation of moment of inertia.*

Equation (3.22) is very useful in the calculation of the moment of inertia utilizing the moment of inertia of known bodies. For example, the moment of inertial of half a circle is half of whole circle for axis a the center of circle. The moment of inertia can then move the center of area. of the

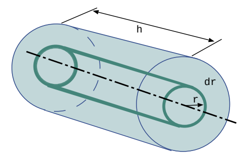



*Fig. -3.6. Cylinder with an element for calculation moment of inertia.*

### 3.3.3 Examples of Moment of Inertia

Example 3.2:
*Calculate the moment of inertia for the mass of the cylinder about center axis which height of $h$ and radius, $r_0$, as shown in Figure 3.6. The material is with an uniform density and homogeneous.*

SOLUTION

The element can be calculated using cylindrical coordinate. Here the convenient element is a shell of thickness $dr$ which shown in Figure 3.6 as

$$I_{rr} = \rho \int_V r^2 dm = \rho \int_0^{r_0} r^2 \overbrace{h\, 2\, \pi\, r\, dr}^{dV} = \rho\, h\, 2\, \pi \frac{{r_0}^4}{4} = \frac{1}{2} \rho h \pi {r_0}^4 = \frac{1}{2} m\, {r_0}^2$$

The radius of gyration is

$$r_k = \sqrt{\frac{\frac{1}{2}\, m {r_0}^2}{m}} = \frac{r_0}{\sqrt{2}}$$

End Solution

Example 3.3:
*Calculate the moment of inertia of the rectangular shape shown in Figure 3.7 around $\mathbf{x}$ coordinate.*

SOLUTION

The moment of inertia is calculated utilizing equation (3.14) as following

$$I_{xx} = \int_A \left( \overbrace{y^2}^{0} + z^2 \right) dA = \int_0^a z^2 \overbrace{b dz}^{dA} = \frac{a^3\, b}{3}$$

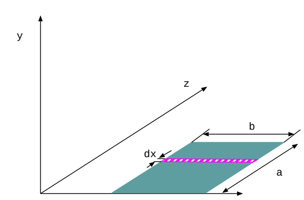



*Fig. -3.7. Description of rectangular in x–y plane for calculation of moment of inertia.*

This value will be used in later examples.

End Solution

Example 3.4:

*To study the assumption of zero thickness, consider a simple shape to see the effects of this assumption. Calculate the moment of inertia about the center of mass of a square shape with a thickness, $t$ compare the results to a square shape with zero thickness.*

SOLUTION

The moment of inertia of transverse slice about $y^{'}$ (see Figure mech:fig:squareEll) is

$$dI_{xxm} = \rho \overbrace{dy}^{t} \overbrace{\frac{b\,a^3}{12}}^{I_{xx}} \tag{3.23}$$

The transformation into from local axis $x$ to center axis, $x^{'}$ can be done as following

$$dI_{x'x'm} = \rho dy \left( \overbrace{\frac{b\,a^3}{12}}^{I_{xx}} + \overbrace{\underbrace{z^2}_{r^2} \; \underbrace{b\,a}_{A}}^{r^2\,A} \right) \tag{3.24}$$

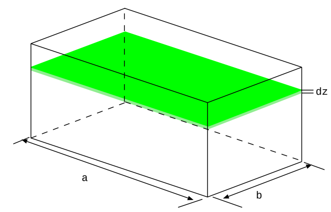



*Fig. -3.8. A square element for the calculations of inertia of two-dimensional to three–dimensional deviations.*

The total moment of inertia can be obtained by integration of equation (3.24) to write as

$$I_{xxm} = \rho \int_{-t/2}^{t/2} \left( \frac{b\,a^3}{12} + z^2\,b\,a \right) dz = \rho\,t\,\frac{a\,b\,t^2 + a^3\,b}{12} \tag{3.25}$$

Comparison with the thin body results in

$$\frac{I_{xx}\,\rho\,t}{I_{xxm}} = \frac{b\,a^3}{t^2\,b\,a + b\,a^3} = \frac{1}{1 + \frac{t^2}{a^2}} \tag{3.26}$$

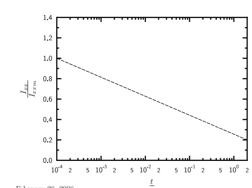



*Fig. -3.9. The ratio of the moment of inertia of two-dimensional to three–dimensional.*

It can be noticed right away that equation (3.26) indicates that ratio approaches one when thickness ratio is approaches zero, $I_{xxm}(t \to 0) \to 1$. Additionally it can be noticed that the ratio $a^2/t^2$ is the only contributor to the error[2]. The results are present in Figure 3.9. I can be noticed that the error is significant very fast even for small values of $t/a$ while the with of the box, $b$ has no effect on the error.

End Solution

[2]This ratio is a dimensionless number that commonly has no special name. This author suggests to call this ratio as the B number.

Example 3.5:

*Calculate the rectangular moment of Inertia for the rotation trough center in $zz$ axis (axis of rotation is out of the page). Hint, construct a small element and build longer build out of the small one. Using this method calculate the entire rectangular.*

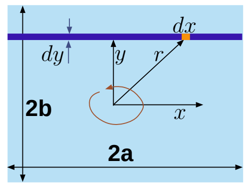



*Fig. -3.10. Rectangular Moment of inertia.*

SOLUTION

The moment of inertia for a long element with a distance $y$ shown in Figure 3.10 is

$$d\,I_{zz}\big|_{dy} = \int_{-a}^{a} \overbrace{\left(y^2+x^2\right)}^{r^2} dy\,dx = \frac{2\,\left(3\,a\,y^2+a^3\right)}{3}\,dy \tag{3.V.a}$$

The second integration ( no need to use (3.20), why?) is

$$I_{zz} = \int_{-b}^{b} \frac{2\,\left(3\,a\,y^2+a^3\right)}{3}\,dy \tag{3.V.b}$$

Results in

$$I_{zz} = \frac{a\,\left(2\,a\,b^3+2\,a^3\,b\right)}{3} = \overbrace{A}^{4\,a\,b}\left(\frac{(2a)^2+(2b)^2}{12}\right) \tag{3.V.c}$$

Or

End Solution

Example 3.6:

*Calculate the center of area and moment of inertia for the parabola, $y=\alpha x^2$, depicted in Figure 3.11. Hint, calculate the area first. Use this area to calculate moment of inertia. There are several ways to approach the calculation (different infinitesimal area).*

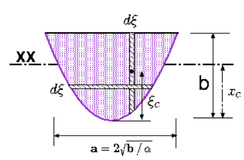



*Fig. -3.11. Parabola for calculations of moment of inertia.*

SOLUTION

For $y=b$ the value of $x=\sqrt{b/\alpha}$. First the area inside the parabola calculated as

$$A = 2\int_0^{\sqrt{b/\alpha}} \overbrace{(b-\alpha\xi^2)d\xi}^{dA/2} = \frac{2(3\,\alpha-1)}{3}\,\left(\frac{b}{\alpha}\right)^{\frac{3}{2}}$$

The center of area can be calculated utilizing equation (3.6). The center of every element is at, $\left(\alpha\,\xi^2 + \frac{b-\alpha\xi^2}{2}\right)$ the element area is used before and therefore

$$x_c = \frac{1}{A}\int_0^{\sqrt{b/\alpha}} \overbrace{\left(\alpha\xi^2 + \frac{(b-\alpha\xi^2)}{2}\right)}^{x_c} \overbrace{(b-\alpha\xi^2)d\xi}^{dA} = \frac{3\,\alpha\,b}{15\,\alpha - 5} \tag{3.27}$$

The moment of inertia of the area about the center can be found using in equation (3.27) can be done in two steps first calculate the moment of inertia in this coordinate system and then move the coordinate system to center. Utilizing equation (3.14) and doing the integration from 0 to maximum y provides

$$I_{x'x'} = 4\int_0^b \xi^2 \overbrace{\sqrt{\frac{\xi}{\alpha}}\,d\xi}^{dA} = \frac{2\,b^{7/2}}{7\,\sqrt{\alpha}}$$

Utilizing equation (3.20)

$$I_{xx} = I_{x'x'} - A\,\Delta x^2 = \overbrace{\frac{4\,b^{7/2}}{7\,\sqrt{\alpha}}}^{I_{x'x'}} - \overbrace{\frac{3\,\alpha - 1}{3}\,\left(\frac{b}{\alpha}\right)^{\frac{3}{2}}}^{A} \overbrace{\left(\frac{3\,\alpha\,b}{15\,\alpha - 5}\right)^2}^{(\Delta x = x_c)^2}$$

or after working the details results in

$$I_{xx} = \frac{\sqrt{b}\,\left(20\,b^3 - 14\,b^2\right)}{35\,\sqrt{\alpha}}$$

End Solution

Example 3.7:

*Calculate the moment of inertia of strait angle triangle about its $y$ axis as shown in the Figure on the right. Assume that base is $\mathbf{a}$ and the height is $\mathbf{h}$. What is the moment when a symmetrical triangle is attached on left. What is the moment when a symmetrical triangle is attached on bottom. What is the moment inertia when $a \longrightarrow 0$. What is the moment inertia when $h \longrightarrow 0$.*

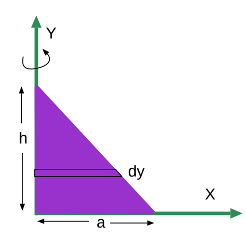



*Fig. -3.12. Triangle for example 3.7.*

SOLUTION

The right edge line equation can be calculated as

$$\frac{y}{h} = \left(1 - \frac{x}{a}\right)$$

or

$$\frac{x}{a} = \left(1 - \frac{y}{h}\right)$$

Now using the moment of inertia of rectangle on the side $(\mathbf{y})$ coordinate (see example (3.3))

$$\int_0^h \frac{a\left(1 - \dfrac{y}{h}\right)^3 dy}{3} = \frac{a^3\, h}{4}$$

For two triangles attached to each other the moment of inertia will be sum as $\frac{a^3\, h}{2}$
The rest is under construction.

End Solution

### 3.3.4 Product of Inertia

In addition to the moment of inertia, the product of inertia is commonly used. Here only the product of the area is defined and discussed. The product of inertia defined as

$$I_{x_i\, x_j} = \int_A x_i\, x_j dA \tag{3.28}$$

For example, the product of inertia for $x$ and $y$ axises is

$$I_{xy} = \int_A x\, y dA \tag{3.29}$$

Product of inertia can be positive or negative value as oppose the moment of inertia. The calculation of the product of inertia isn't different much for the calculation of the moment of inertia. The units of the product of inertia are the same as for moment of inertia.

#### Transfer of Axis Theorem

Same as for moment of inertia there is also similar theorem.

$$I_{x'\,y'} = \int_A x^{'}\, y^{'} dA = \int_A (x + \Delta x)\,(y + \Delta y) dA \tag{3.30}$$

expanding equation (3.30) results in

$$I_{x'\,y'} = \overbrace{\int_A x\, y dA}^{I_{xy}} + \overbrace{\int_A x\, \Delta y dA}^{\Delta y \overbrace{\int_A x\, dA}^{0}} + \overbrace{\int_A \Delta x\, y dA}^{\Delta x \overbrace{\int_A y\, dA}^{0}} + \overbrace{\int_A \Delta x\, \Delta y dA}^{\Delta x\, \Delta y\, A} \tag{3.31}$$

The final form is

$$I_{x'y'} = I_{xy} + \Delta x\, \Delta y\, A \tag{3.32}$$

There are several relationships should be mentioned

$$I_{xy} = I_{yx} \tag{3.33}$$

Symmetrical area has zero product of inertia because integration of odd function (asymmmertial function) left part cancel the right part.

Example 3.8:
*Calculate the product of inertia of straight edge triangle.*

SOLUTION

The equation of the line is

$$y = \frac{a}{b}x + a$$

The product of inertia at the center is zero. The total product of inertia is

$$I_{x'y'} = 0 + \overbrace{\frac{a}{3}}^{\Delta x}\,\overbrace{\frac{b}{3}}^{\Delta y}\,\overbrace{\left(\frac{a\,b}{2}\right)}^{A} = \frac{a^2\,b^2}{18}$$

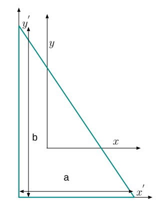



*Fig. -3.13. Product of inertia for triangle.*

End Solution

### 3.3.5 Principal Axes of Inertia

The inertia matrix or inertia tensor is

$$\begin{vmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{vmatrix} \tag{3.34}$$

In linear algebra it was shown that for some angle equation (3.34) can be transform into

$$\begin{vmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{vmatrix} \tag{3.35}$$

System which creates equation (3.35) referred as principle system.

## 3.4 Newton's Laws of Motion

These laws can be summarized in two statements one, for every action by body **A** on Body **B** there is opposite reaction by body **B** on body **A**. Two, which can expressed in mathematical form as

$$\sum \mathbf{F} = \frac{D\,(m\,U)}{Dt} \tag{3.36}$$

It can be noted that $D$ replaces the traditional $d$ since the additional meaning which be added. Yet, it can be treated as the regular derivative. This law apply to any body and any body can "broken" into many small bodies which connected to each other. These small "bodies" when became small enough equation (3.36) can be transformed to a continuous form as

$$\sum \mathbf{F} = \int_V \frac{D\ (\rho\,U)}{Dt} dV \tag{3.37}$$

The external forces are equal to internal forces the forces between the "small" bodies are cancel each other. Yet this examination provides a tool to study what happened in the fluid during operation of the forces.

Since the derivative with respect to time is independent of the volume, the derivative can be taken out of the integral and the alternative form can be written as

$$\sum \mathbf{F} = \frac{D}{Dt} \int_V \rho\,U dV \tag{3.38}$$

The velocity, $U$ is a derivative of the location with respect to time, thus,

$$\sum \mathbf{F} = \frac{D^2}{Dt^2} \int_V \rho\,r dV \tag{3.39}$$

where $r$ is the location of the particles from the origin.

The external forces are typically divided into two categories: body forces and surface forces. The body forces are forces that act from a distance like magnetic field or gravity. The surface forces are forces that act on the surface of the body (pressure, stresses). The same as in the dynamic class, the system acceleration called the internal forces. The acceleration is divided into three categories: Centrifugal, $\boldsymbol{\omega}\times(\mathbf{r}\times\boldsymbol{\omega})$, Angular, $\mathbf{r}\times\dot{\boldsymbol{\omega}}$, Coriolis, $\mathbf{2}\,(\mathbf{U_r}\times\boldsymbol{\omega})$. The radial velocity is denoted as $U_r$.

## 3.5 Angular Momentum and Torque

The angular momentum of body, $dm$, is defined as

$$L = \mathbf{r}\times\mathbf{U}dm \tag{3.40}$$

The angular momentum of the entire system is calculated by integration (summation) of all the particles in the system as

$$L_s = \int_m \mathbf{r}\times U\,dm \tag{3.41}$$

The change with time of angular momentum is called torque, in analogous to the momentum change of time which is the force.

$$T_\tau = \frac{D\,L}{Dt} = \frac{D}{Dt}\left(\mathbf{r}\times\mathbf{U}dm\right) \tag{3.42}$$

where $T_\tau$ is the torque. The torque of entire system is

$$T_{\tau s} = \int_m \frac{D\,L}{Dt} = \frac{D}{Dt}\int_m \left(\mathbf{r}\times\mathbf{U}dm\right) \tag{3.43}$$

It can be noticed (well, it can be proved utilizing vector mechanics) that

$$T_\tau = \frac{D}{Dt}(\mathbf{r}\times\mathbf{U}) = \frac{D}{Dt}(\mathbf{r}\times\frac{D\,r}{Dt}) = \frac{D^2\mathbf{r}}{Dt^2} \tag{3.44}$$

To understand these equations a bit better, consider a particle moving in x–y plane. A force is acting on the particle in the same plane (x–y) plane. The velocity can be written as $U = u\hat{\imath} + v\hat{\jmath}$ and the location from the origin can be written as $\mathbf{r} = x\hat{\imath} + y\hat{\jmath}$. The force can be written, in the same fashion, as $\mathbf{F} = F_x\hat{\imath} + F_y\hat{\jmath}$. Utilizing equation (3.40) provides

$$\mathbf{L} = \mathbf{r}\times\mathbf{U} = \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & 0 \\ u & v & 0 \end{pmatrix} = (x\,v - y\,u)\hat{k} \tag{3.45}$$

Utilizing equation (3.42) to calculate the torque as

$$T_\tau = \mathbf{r}\times\mathbf{F} = \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & 0 \\ F_x & F_y & 0 \end{pmatrix} = (x\,F_x - y\,F_y)\hat{k} \tag{3.46}$$

Since the torque is a derivative with respect to the time of the angular momentum it is also can be written as

$$xF_x - yF_y = \frac{D}{Dt}\left[(xv - yu)\,dm\right] \tag{3.47}$$

The torque is a vector and the various components can be represented as

$$T_{\tau x} = \hat{\imath}\bullet\frac{D}{Dt}\int_m \mathbf{r}\times\mathbf{U}\,dm \tag{3.48}$$

In the same way the component in $y$ and $z$ can be obtained.

### 3.5.1 Tables of geometries

Th following tables present several moment of inertias of commonly used geometries.

*Table -3.1. Moments of Inertia for various plane surfaces about their center of gravity (full shapes)*

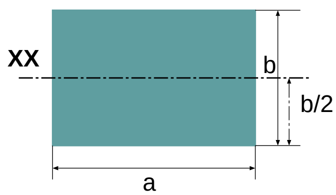
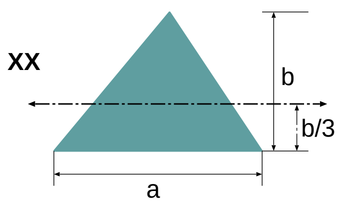
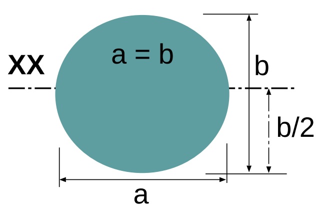
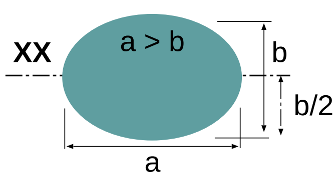
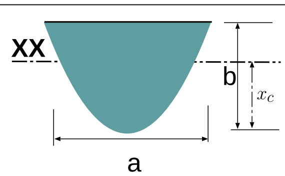
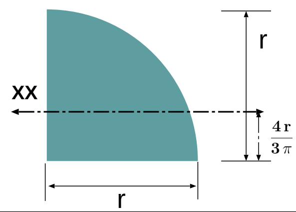
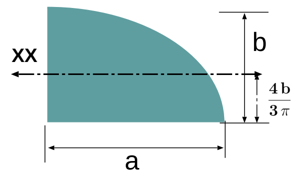
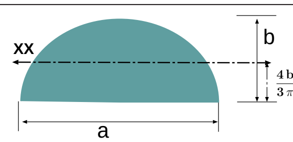
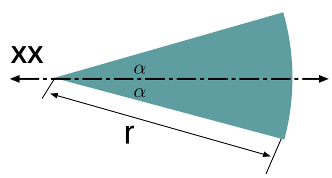
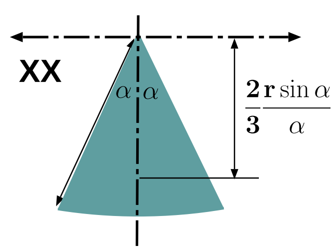
| Shape Name | Picture description | $x_c$, $y_c$ | A | $I_{xx}$ |
|---|---|---|---|---|
| Rectangle |   | $\frac{b}{2}\,;\,\frac{a}{2}$ | $a\,b$ | $\frac{ab^3}{12}$ |
| Triangle |   | $\frac{a}{3}$ | $\frac{a\,b}{3}$ | $\frac{ab^3}{36}$ |
| Circle |   | $\frac{b}{2}$ | $\frac{\pi\,b^2}{4}$ | $\frac{\pi b^4}{64}$ |
| Ellipse |   | $\frac{b}{2}\;\frac{b}{2}$ | $\frac{\pi\,ab}{4}$ | $\frac{Ab^2}{64}$ |
| $y = \alpha x^2$ Parabola |   | $\frac{3\,\alpha\,b}{15\,\alpha-5}$ | $\frac{6\alpha-2}{3}\times\left(\frac{b}{\alpha}\right)^{\frac{3}{2}}$ | $\frac{\sqrt{b}\left(20\,b^3-14\,b^2\right)}{35\,\sqrt{\alpha}}$ |

*Table -3.2. Moment of inertia for various plane surfaces about their center of gravity*

| Shape Name | Picture description | $x_c,\ y_c$ | A | $I_{xx}$ |
|---|---|---|---|---|
| Quadrant of Circle |   | $\dfrac{4\,r}{3\,\pi}$ | $\dfrac{\pi\,r^2}{4}$ | $r^4\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)$ |
| Ellipsoidal Quadrant |   | $\dfrac{4\,b}{3\,\pi}$ | $\dfrac{\pi\,a\,b}{4}$ | $a\,b^3\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)$ |
| Half of Elliptic |   | $\dfrac{4\,b}{3\,\pi}$ | $\dfrac{\pi\,a\,b}{4}$ | $a\,b^3\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)$ |
| Circular Sector |   | $0$ | $2\alpha\,r^2$ | $\frac{r^4}{4}\left(\alpha-\frac{1}{2}\sin 2\alpha\right)$ |
| Circular Sector |   | $\frac{2}{3}\,\frac{r\,\sin\alpha}{\alpha}$ | $2\alpha\,r^2$ | $I_{x'x'} =$ $\frac{r^4}{4}\left(\alpha+\frac{1}{2}\sin 2\alpha\right)$ |

# CHAPTER 4

# Fluids Statics

## 4.1 Introduction

The simplest situation that can occur in the study of fluid is when the fluid is at rest or quasi rest. This topic was introduced to most students in previous study of rigid body. However, here this topic will be more vigorously examined. Furthermore, the student will be exposed to stability analysis probably for the first time. Later, the methods discussed here will be expanded to more complicated dynamics situations.

## 4.2 The Hydrostatic Equation

A fluid element with dimensions of $DC$, $dy$, and $dz$ is motionless in the accelerated system, with acceleration, $a$ as shown in Figure 4.1. The system is in a body force field, $g_G(x, y, z)$. The combination of an acceleration and the body force results in effective body force which is

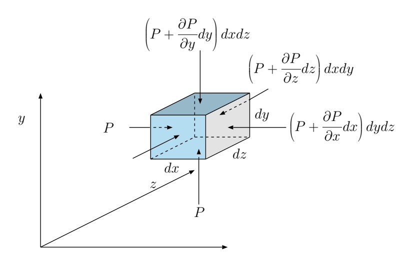



*Fig. -4.1. Description of a fluid element in accelerated system under body forces.*

$$\mathbf{g}_{\mathsf{G}} - a = g_{\mathsf{eff}} \tag{4.1}$$

Equation (4.1) can be reduced and simplified for the case of no acceleration, $\mathbf{a} = 0$.

In these derivations, several assumptions must be made. The first assumption is that the change in the pressure is a continuous function. There is no requirement that the pressure has to be a monotonous function e.g. that pressure can increase and later decrease. The changes of the second derivative pressure are not significant compared to the first derivative ($\partial P/\partial n \times d\ell >> \partial^2 P/\partial n^2$). where $n$ is the steepest direction of the pressure derivative and $d\ell$ is the infinitesimal length. This mathematical statement simply requires that the pressure can deviate in such a way that the average on infinitesimal area can be found and expressed as only one direction. The net pressure force on the faces in the $x$ direction results in

$$d\mathbf{F} = -\left(\frac{\partial P}{\partial x}\right) dydx\, \hat{i} \tag{4.2}$$

In the same fashion, the calculations of the three directions result in the total net pressure force as

$$\sum_{\text{surface}} F = -\left(\frac{\partial P}{\partial x}\,\hat{i} + \frac{\partial P}{\partial y}\,\hat{j} + \frac{\partial P}{\partial y}\,\hat{k}\right) \tag{4.3}$$

The term in the parentheses in equation (4.3) referred to in the literature as the pressure gradient (see for more explanation in the Mathematics Appendix). This mathematical operation has a geometrical interpretation. If the pressure, $P$, was a two–dimensional height (that is only a function of $x$ and $y$) then the gradient is the steepest ascent of the height (to the valley). The second point is that the gradient is a vector (that is, it has a direction). Even though, the pressure is treated, now, as a scalar function (there no reference to the shear stress in part of the pressure) the gradient is a vector. For example, the dot product of the following is

$$\widehat{i} \cdot \mathbf{grad} P = \widehat{i} \cdot \nabla P = \frac{\partial P}{\partial x} \tag{4.4}$$

In general, if the coordinates were to "rotate/transform" to a new system which has a different orientation, the dot product results in

$$\overline{i_n} \cdot \mathbf{grad} P = \overline{i_n} \cdot \nabla P = \frac{\partial P}{\partial n} \tag{4.5}$$

where $i_n$ is the unit vector in the $n$ direction and $\partial/\partial n$ is a derivative in that direction.

As before, the effective gravity force is utilized in case where the gravity is the only body force and in an accelerated system. The body (element) is in rest and therefore the net force is zero

$$\sum_{\text{total}} \mathbf{F} = \sum_{\text{surface}} \mathbf{F} + \sum_{\text{body}} \mathbf{F} \tag{4.6}$$

Hence, the utilizing the above derivations one can obtain

$$-\mathbf{grad} P dx\, dy\, dz + \rho\, g_{\text{eff}} dx\, dy\, dz = 0 \tag{4.7}$$

or

Pressure Gradient

$$\boxed{\mathbf{grad} P = \nabla P = \rho\, g_{\text{eff}}} \tag{4.8}$$

Some refer to equation (4.8) as the Fluid Static Equation. This equation can be integrated and therefore solved. However, there are several physical implications to this equation which should be discussed and are presented here. First, a discussion on a simple condition and will continue in more challenging situations.

## 4.3 Pressure and Density in a Gravitational Field

In this section, a discussion on the pressure and the density in various conditions is presented.

### 4.3.1 Constant Density in Gravitational Field

The simplest case is when the density, $\rho$, pressure, $P$, and temperature, $T$ (in a way no function of the location) are constant. Traditionally, the $z$ coordinate is used as the (negative) direction of the gravity[1]. The effective body force is

$$g_{\text{eff}} = -\mathbf{g}\,\hat{k} \tag{4.9}$$

Utilizing equation (4.9) and substituting it into equation (4.8) results into three simple partial differential equations. These equations are

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0 \tag{4.10}$$

and

Pressure Change

$$\boxed{\frac{\partial P}{\partial z} = -\rho\,\mathbf{g}} \tag{4.11}$$

Equations (4.10) can be integrated to yield

$$P(x, y) = constant \tag{4.12}$$

and constant in equation (4.12) can be absorbed by the integration of equation (4.11) and therefore

$$P(x, y, z) = -\rho g z + constant \tag{4.13}$$

The integration constant is determined from the initial conditions or another point. For example, if at point $z_0$ the pressure is $P_0$ then the equation (4.13) becomes

$$P(z) - P_0 = -\rho g (z - z_0) \tag{4.14}$$

[1]This situation were the tradition is appropriated, it will be used. There are fields where $x$ or $y$ are designed to the direction of the gravity and opposite direction. For this reason sometime there will be a deviation from the above statement.

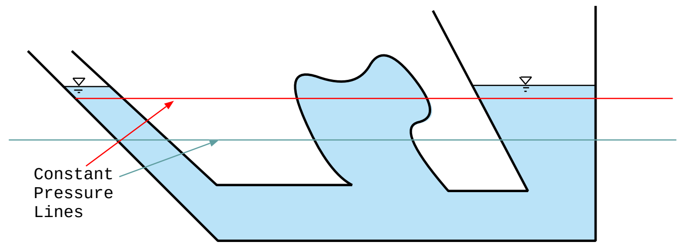



*Fig. -4.2. Pressure lines in a static fluid with a constant density.*

It is evident from equation (4.13) that the pressure depends only on $z$ and/or the constant pressure lines are in the plane of $x$ and $y$. Figure 4.2 describes the constant pressure lines in the container under the gravity body force. The pressure lines are continuous even in area where there is a discontinuous fluid. The reason that a solid boundary doesn't break the continuity of the pressure lines is because there is always a path to some of the planes.

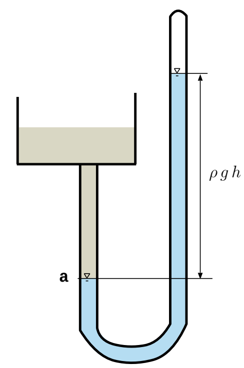



*Fig. -4.3. A schematic to explain the measure of the atmospheric pressure.*

It is convenient to reverse the direction of $z$ to get rid of the negative sign and to define $h$ as the dependent of the fluid that is $h \equiv -(z - z_0)$ so equation (4.14) becomes

Pressure relationship

$$P(h) - P_0 = \rho g h \tag{4.15}$$

In the literature, the right hand side of the equation (4.15) is defined as piezometric pressure.

Example 4.1:

*Two chambers tank depicted in Figure 4.4 are in equilibration. If the air mass at chamber A is 1 Kg while the mass at chamber B is unknown. The difference in the* *liquid heights between the two chambers is $2[m]$. The liquid in the two chambers is water. The area of each chamber is $1[m^2]$. Calculate the air mass in chamber B. You can assume ideal gas for the air and the water is incompressible substance with density of $1000[kg/m^2]$. The total height of the tank is $4[m]$. Assume that the chamber are at the same temperature of 27°C.*

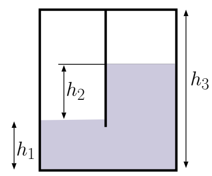



*Fig. -4.4. The effective gravity is for accelerated cart.*

Solution

The equation of state for the chamber A is

$$m_A = \frac{R\,T}{P_A\,V_A} \tag{4.I.a}$$

The equation of state for the second chamber is

$$m_B = \frac{R\,T}{P_B\,V_B} \tag{4.I.b}$$

The water volume is

$$V_{total} = h_1\,A + (h_1 + h_2)A = (2\,h_1 + h_2)\,A \tag{4.I.c}$$

The pressure difference between the liquid interface is estimated negligible the air density as

$$P_A - P_B = \Delta P = h_2\,\rho\,g \tag{4.I.d}$$

combining equations (4.I.a), (4.I.b) results in

$$\frac{R\,T}{m_A\,V_A} - \frac{R\,T}{m_B\,V_B} = h_2\,\rho\,g \Longrightarrow \left(1 - \frac{1}{\dfrac{m_B}{m_A}\,\dfrac{V_B}{V_A}}\right) = \frac{h_2\,\rho\,g\,m_A\,V_A}{R\,T} \tag{4.I.e}$$

In equation the only unknown is the ratio of $m_B/m_A$ since everything else is known. Denoting $X = m_B/m_A$ results in

$$\frac{1}{X} = 1 - \frac{h_2\,\rho\,g\,m_A\,V_A}{R\,T} \Longrightarrow X = \frac{1}{1 - \dfrac{h_2\,\rho\,g\,m_A\,V_A}{R\,T}} \tag{4.I.f}$$

End Solution

The following question is a very nice qualitative question of understanding this concept.

Example 4.2:

*A tank with opening at the top to the atmosphere contains two immiscible liquids one heavy and one light as depicted in Figure 4.5 (the light liquid is on the top of the heavy liquid). Which piezometric tube will be higher? why? and how much higher? What is the pressure at the bottom of the tank?*

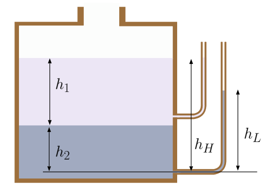



*Fig. -4.5. Tank and the effects different liquids.*

SOLUTION

The common instinct is to find that the lower tube will contain the higher liquids. For the case, the lighter liquid is on the top the heavier liquid the the top tube is the same as the surface. However, the lower tube will raise only to (notice that $g$ is canceled)

$$h_L = \frac{\rho_1\, h_1 + \rho_2\, h_2}{\rho_2} \tag{4.II.a}$$

Since $\rho_1 > \rho_1$ the mathematics dictate that the height of the second is lower. The difference is

$$\frac{h_H - h_L}{h_2} = \frac{h_H}{h_2} - \left(\frac{\rho_1\, h_1 + \rho_2\, h_2}{h_r 21\, \rho_2}\right) \tag{4.II.b}$$

It can be noticed that $h_H = h_1 + h - 2$ hence,

$$\frac{h_H - h_L}{h_2} = \frac{h_1 + h_2}{h_2} - \left(\frac{\rho_1\, h_1 + \rho_2\, h_2}{h_2\, \rho_2}\right) = \frac{h_1}{h_2}\left(1 - \frac{\rho_1}{\rho_2}\right) \tag{4.II.c}$$

or

$$h_H - h_L = h_1\left(1 - \frac{\rho_1}{\rho_2}\right) \tag{4.II.d}$$

The only way the $h_L$ to be higher of $h_H$ is if the heavy liquid is on the top if the stability allow it. The pressure at the bottom is

$$P = P_{atmos} + g\,(\rho_1\, h_1 + \rho_2\, h_2) \tag{4.16}$$

End Solution

Example 4.3:

*The effect of the water in the car tank is more than the possibility that water freeze in fuel lines. The water also can change measurement of fuel gage. The way the interpretation of an automobile fuel gage is proportional to the pressure at the bottom of the fuel tank. Part of the tank height is filled with the water at the bottom (due to the larger density). Calculate the error for a give ratio between the fuel density to the water.*

SOLUTION

The ratio of the fuel density to water density is $\varsigma = \rho_f / \rho_w$ and the ratio of the total height to the water height is $x = h_w / h_{total}$ Thus the pressure at the bottom when the tank is full with only fuel

$$P_{full} = \rho_f \, h_{total} \, g \tag{4.III.a}$$

But when water is present the pressure will be the same at

$$P_{full} = \left( \rho_w \, x \, + \phi \, \rho_f \right) g \, h_{total} \tag{4.III.b}$$

and if the two are equal at

$$\rho_f \, \cancel{h_{total}} \, \cancel{g} = \left( \rho_w \, x \, + \phi \, \rho_f \right) \cancel{g} \, \cancel{h_{total}} \tag{4.III.c}$$

where $\phi$ in this case the ratio of the full height (on the fake) to the total height. Hence,

$$\phi = \frac{\rho_f - x \, \rho_w}{\rho_f} \tag{4.III.d}$$

End Solution

## 4.3.2 Pressure Measurement

### 4.3.2.1 Measuring the Atmospheric Pressure

One of the application of this concept is the idea of measuring the atmospheric pressure. Consider a situation described in Figure 4.3. The liquid is filling the tube and is brought into a steady state. The pressure above the liquid on the right side is the vapor pressure. Using liquid with a very low vapor pressure like mercury, will result in a device that can measure the pressure without additional information (the temperature).

Example 4.4:

*Calculate the atmospheric pressure at $20°C$. The high of the Mercury is 0.76 [m] and the gravity acceleration is 9.82[$m/sec$]. Assume that the mercury vapor pressure is 0.000179264[kPa]. The description of the height is given in Figure 4.3. The mercury density is 13545.85[$kg/m^3$].*

SOLUTION

The pressure is uniform or constant plane perpendicular to the gravity. Hence, knowing any point on this plane provides the pressure anywhere on the plane. The atmospheric pressure at point **a** is the same as the pressure on the right hand side of the tube. Equation (4.15) can be utilized and it can be noticed that pressure at point **a** is

$$P_a = \rho \, g \, h + P_{vapor} \tag{4.17}$$

The density of the mercury is given along with the gravity and therefore,

$$P_a = 13545.85 \times 9.82 \times 0.76 \sim 101095.39[Pa] \sim 1.01[Bar]$$

The vapor pressure is about $1 \times 10^{-4}$ percent of the total results.

End Solution

The main reason the mercury is used because of its large density and the fact that it is in a liquid phase in most of the measurement range. The third reason is the low vapor (partial) pressure of the mercury. The partial pressure of mercury is in the range of the 0.000001793[Bar] which is insignificant compared to the total measurement as can be observed from the above example.

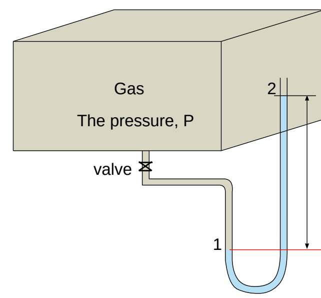



*Fig. -4.6. Schematic of gas measurement utilizing the "U" tube.*

Example 4.5:

*A liquid[2] **a** in amount $H_a$ and a liquid **b** in amount $H_b$ in to an U tube. The ratio of the liquid densities is $\alpha = \rho_1/\rho_2$. The width of the U tube is $L$. Locate the liquids surfaces.*

Solution

The question is to find the equilibrium point where two liquids balance each other. If the width of the U tube is equal or larger than total length of the two liquids then the whole liquid will be in bottom part. For smaller width, $L$, the ratio between two sides will be as

$$\rho_1 \, h_1 = \rho_2 \, h_2 \rightarrow h_2 = \alpha \, h_1$$

The mass conservation results in

$$H_a + H_b = L + h_1 + h_2$$

Thus two equations and two unknowns provide the solution which is

$$h_1 = \frac{H_a + H_b - L}{1 + \alpha}$$

When $H_a > L$ and $\rho_a \, (H_a - L) \geq \rho_b$ (or the opposite) the liquid $\mathbf{a}$ will be on the two sides of the U tube. Thus, the balance is

$$h_1 \, \rho_b + h_2 \, \rho_a = h_3 \, \rho_a$$

where $h_1$ is the height of liquid $\mathbf{b}$ where $h_2$ is the height of "extra" liquid $\mathbf{a}$ and same side as liquid $\mathbf{b}$ and where $h_3$ is the height of liquid $\mathbf{b}$ on the other side. When in this case $h_1$ is equal to $H_b$. The additional equation is the mass conservation as

$$H_a = h_2 + L + h_3$$

[2]This example was requested by several students who found their instructor solution unsatisfactory.

The solution is

$$h_2 = \frac{(H_a - L)\,\rho_a - H_b \rho_b}{2\,\rho_a}$$

End Solution

#### 4.3.2.2 Pressure Measurement

The idea describes the atmospheric measurement that can be extended to measure the pressure of the gas chambers. Consider a chamber filled with gas needed to be measured (see Figure 4.6). One technique is to attached "U" tube to the chamber and measure the pressure. This way, the gas is prevented from escaping and its pressure can be measured with a minimal interference to the gas (some gas enters to the tube).

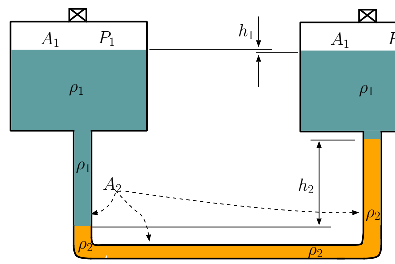



*Fig. -4.7. Schematic of sensitive measurement device.*

The gas density is significantly lower than the liquid density and therefore can be neglected. The pressure at point "1" is

$$P_1 = P_{atmos} + \rho g\, h \tag{4.18}$$

Since the atmospheric pressure was measured previously (the technique was shown in the previous section) the pressure of the chamber can be measured.

#### 4.3.2.3 Magnified Pressure Measurement

For situations where the pressure difference is very small, engineers invented more sensitive measuring device. This device is build around the fact that the height is a function of the densities difference. In the previous technique, the density of one side was neglected (the gas side) compared to other side (liquid). This technique utilizes the opposite range. The densities of the two sides are very close to each other, thus the height become large. Figure 4.7 shows a typical and simple schematic of such an instrument. If the pressure differences between $P_1$ and $P_2$ is small this instrument can "magnified" height, $h_1$ and provide "better" accuracy reading. This device is based on the following mathematical explanation.

In steady state, the pressure balance (only differences) is

$$P_1 + g\,\rho_1 (h_1 + h_2) = P_2 + g\,h_2\,\rho_2 \tag{4.19}$$

It can be noticed that the "missing height" is canceled between the two sides. It can be noticed that $h_1$ can be positive or negative or zero and it depends on the ratio that two containers filled with the light density liquid. Additionally, it can be observed that $h_1$ is relatively small because $A_1 >> A_2$. The densities of the liquid are chosen so that they are close to each other but not equal. The densities of the liquids are chosen to be much heavier than the measured gas density. Thus, in writing equation (4.19) the gas density was neglected. The pressure difference can be expressed as

$$P_1 - P_2 = g\left[\rho_2\, h_2 - \rho_1 (h_1 + h_2)\right] \tag{4.20}$$

If the light liquid volume in the two containers is known, it provides the relationship between $h_1$ and $h_2$. For example, if the volumes in two containers are equal then

$$-h_1\, A_1 = h_2\, A_2 \longrightarrow h_1 = -\frac{h_2\, A_2}{A_1} \tag{4.21}$$

Liquid volumes do not necessarily have to be equal. Additional parameter, the volume ratio, will be introduced when the volumes ratio isn't equal. The calculations as results of this additional parameter does not cause a significant complications. Here, this ratio equals to one and it simplify the equation (4.21). But this ratio can be inserted easily into the derivations. With the equation for height (4.21) equation (4.19) becomes

$$P_1 - P_2 = g\, h_2 \left(\rho_2 - \rho_1 \left(1 - \frac{A_2}{A_1}\right)\right) \tag{4.22}$$

or the height is

$$h_2 = \frac{P_1 - P_2}{g\left[(\rho_2 - \rho_1) + \rho_1 \frac{A_2}{A_1}\right]} \tag{4.23}$$

For the small value of the area ratio, $A_2/A_1 << 1$, then equation (4.23) becomes

$$h_2 = \frac{P_1 - P_2}{g\,(\rho_2 - \rho_1)} \tag{4.24}$$

Some refer to the density difference shown in equation (4.24) as "magnification factor" since it replace the regular density, $\rho_2$.

**Inclined Manometer**

One of the old methods of pressure measurement is the inclined manometer. In this method, the tube leg is inclined relatively to gravity (depicted in Figure 4.8). This method is an attempt to increase the accuracy by "extending" length visible of the tube. The equation (4.18) is then

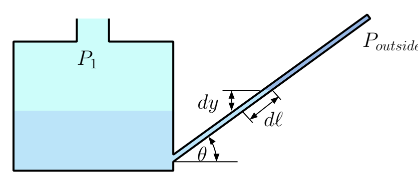



*Fig. -4.8. Inclined manometer.*

$$P_1 - P_{outside} = \rho\, g\, d\ell \tag{4.25}$$

If there is a insignificant change in volume (the area ratio between tube and inclined leg is significant), a location can be calibrated on the inclined leg as zero[3].

[3]This author's personal experience while working in a ship that use this manometer which is significantly inaccurate (first thing to be replaced on the ship). Due to surface tension, caused air entrapment especially in rapid change of the pressure or height.

**Inverted U-tube manometer**

The difference in the pressure of two different liquids is measured by this manometer. This idea is similar to "magnified" manometer but in reversed. The pressure line are the same for both legs on line $ZZ$. Thus, it can be written as the pressure on left is equal to pressure on the right legs (see Figure 4.9).

$$\overbrace{P_2 - \rho_2\,(b+h)}^{\text{right leg}}\; g = \overbrace{P_1 - \rho_1\,a - \rho\,h)}^{\text{left leg}}\; g \tag{4.26}$$

Rearranging equation (4.26) leads to

$$P_2 - P_1 = \rho_2\,(b+h)\,g - \rho_1\,a\,g - \rho\,h\,g \tag{4.27}$$

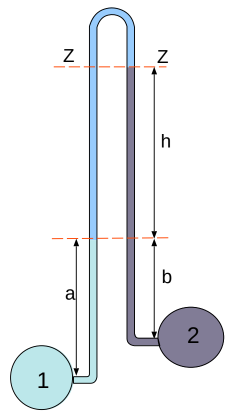


*Fig. -4.9. Schematic of inverted manometer.*

For the similar density of $\rho_1 = \rho_2$ and for $a = b$ equation (4.27) becomes

$$P_2 - P_1 = (\rho_1 - \rho)\,g\,h \tag{4.28}$$

As in the previous "magnified" manometer if the density difference is very small the height become very sensitive to the change of pressure.

### 4.3.3 Varying Density in a Gravity Field

There are several cases that will be discussed here which are categorized as gases, liquids and other. In the gas phase, the equation of state is simply the ideal gas model or the ideal gas with the compressibility factor (sometime referred to as real gas). The equation of state for liquid can be approximated or replaced by utilizing the bulk modulus. These relationships will be used to find the functionality between pressure, density and location.

#### 4.3.3.1 Gas Phase under Hydrostatic Pressure

**Ideal Gas under Hydrostatic Pressure**

The gas density vary gradually with the pressure. As first approximation, the ideal gas model can be employed to describe the density. Thus equation (4.11) becomes

$$\frac{\partial P}{\partial z} = -\frac{g\,P}{R\,T} \tag{4.29}$$

Separating the variables and changing the partial derivatives to full derivative (just a notation for this case) results in

$$\frac{dP}{P} = -\frac{g\,dz}{R\,T} \tag{4.30}$$

Equation (4.30) can be integrated from point "0" to any point to yield

$$\ln \frac{P}{P_0} = -\frac{g}{R\,T}\,(z - z_0) \tag{4.31}$$

It is convenient to rearrange equation (4.31) to the following

$$\frac{P}{P_0} = e^{-\left(\frac{g(z-z_o)}{R\,T}\right)} \tag{4.32}$$

Here the pressure ratio is related to the height exponentially. Equation (4.32) can be expanded to show the difference to standard assumption of constant pressure as

$$\frac{P}{P_0} = 1 - \overbrace{\frac{(z - z_0)\,g}{R\,T}}^{-\frac{h\,\rho_0\,g}{P_0}} + \frac{(z - z_0)^2\,g}{6\,R\,T} + \cdots \tag{4.33}$$

Or in a simplified form where the transformation of $h = (z - z_0)$ to be

$$\frac{P}{P_0} = 1 + \frac{\rho_0\,g}{P_0}\left(h - \overbrace{\frac{h^2}{6} + \cdots}^{\text{correction factor}}\right) \tag{4.34}$$

Equation (4.34) is useful in mathematical derivations but should be ignored for practical use[4].

**Real Gas under Hydrostatic Pressure**

The mathematical derivations for ideal gas can be reused as a foundation for the real gas model ($P = Z\rho RT$). For a large range of $P/P_c$ and $T/T_c$, the value of the compressibility factor, $Z$, can be assumed constant and therefore can be swallowed into equations (4.32) and (4.33). The compressibility is defined in equation (2.39). The modified equation is

$$\frac{P}{P_0} = e^{-\left(\frac{g\,(z-z_o)}{Z\,R\,T}\right)} \tag{4.35}$$

Or in a series form which is

$$\frac{P}{P_0} = 1 - \frac{(z - z_0)\,g}{Z\,R\,T} + \frac{(z - z_0)^2\,g}{6\,Z\,R\,T} + \cdots \tag{4.36}$$

Without going through the mathematics, the first approximation should be noticed that the compressibility factor, $Z$ enter the equation as $h/Z$ and not just $h$. Another point that is worth discussing is the relationship of Z to other gas properties. In general, the relationship is very complicated and in some ranges $Z$ cannot be assumed constant. In these cases, a numerical integration must be carried out.

[4]These derivations are left for a mathematical mind person. These deviations have a limited practical purpose. However, they are presented here for students who need to answer questions on this issue.

#### 4.3.3.2 Liquid Phase Under Hydrostatic Pressure

The bulk modulus was defined in equation (1.28). The simplest approach is to assume that the bulk modulus is constant (or has some representative average). For these cases, there are two differential equations that needed to be solved. Fortunately, here, only one hydrostatic equation depends on density equation. So, the differential equation for density should be solved first. The governing differential density equation (see equation (1.28)) is

$$\rho = B_T \frac{\partial \rho}{\partial P} \tag{4.37}$$

The variables for equation (4.37) should be separated and then the integration can be carried out as

$$\int_{P_0}^{P} d\,P = \int_{\rho_0}^{\rho} B_T \frac{d\rho}{\rho} \tag{4.38}$$

The integration of equation (4.38) yields

$$P - P_0 = B_T \ln \frac{\rho}{\rho_0} \tag{4.39}$$

Equation (4.39) can be represented in a more convenient form as

Density variation

$$\rho = \rho_0 \mathsf{e}^{\frac{P-P_0}{B_T}} \tag{4.40}$$

Equation (4.40) is the counterpart for the equation of state of ideal gas for the liquid phase. Utilizing equation (4.40) in equation (4.11) transformed into

$$\frac{\partial P}{\partial z} = -g\rho_0 \mathsf{e}^{\frac{P-P_0}{B_T}} \tag{4.41}$$

Equation (4.41) can be integrated to yield

$$\frac{B_T}{g\,\rho_0} \mathsf{e}^{\frac{P-P_0}{B_T}} = z + Constant \tag{4.42}$$

It can be noted that $B_T$ has units of pressure and therefore the ratio in front of the exponent in equation (4.42) has units of length. The integration constant, with units of length, can be evaluated at any specific point. If at $z = 0$ the pressure is $P_0$ and the density is $\rho_0$ then the constant is

$$Constant = \frac{B_T}{g\,\rho_0} \tag{4.43}$$

This constant, $B_T/g\,\rho_0$, is a typical length of the problem. Additional discussion will be presented in the dimensionless issues chapter (currently under construction). The solution becomes

$$\frac{B_T}{g\,\rho_0}\left(\mathsf{e}^{\frac{P-P_0}{B_T}}-1\right)=z \tag{4.44}$$

Or in a dimensionless form

Density in Liquids

$$\boxed{\left(\mathsf{e}^{\frac{P-P_0}{B_T}}-1\right)=\frac{z\,g\,\rho_0}{B_T}} \tag{4.45}$$

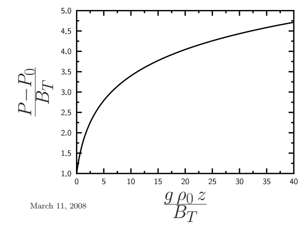



*Fig. -4.10. Hydrostatic pressure when there is compressibility in the liquid phase.*

The solution is presented in equation (4.44) and is plotted in Figure 4.10. The solution is a reverse function (that is not $P = f(z)$ but z = f (P)) it is a monotonous function which is easy to solve for any numerical value (that is only one $z$ corresponds to any Pressure). Sometimes, the solution is presented as

$$\frac{P}{P_0}=\frac{B_T}{P_0}\ln\left(\frac{g\,\rho_0 z}{B_T}+1\right)+1 \tag{4.46}$$

An approximation of equation (4.45) is presented for historical reasons and in order to compare the constant density assumption. The exponent can be expanded as

$$\left(\overbrace{(P-P_0)}^{\text{piezometric pressure}}+\overbrace{\frac{B_T}{2}\left(\frac{P-P_0}{B_T}\right)^2+\frac{B_T}{6}\left(\frac{P-P_0}{B_T}\right)^3+\cdots}^{\text{corrections}}\right)=z\,g\,\rho_0 \tag{4.47}$$

It can be noticed that equation (4.47) is reduced to the standard equation when the normalized pressure ratio, $P/B_T$ is small ($<<1$). Additionally, it can be observed that the correction is on the left hand side and not as the "traditional" correction on the piezometric pressure side.

In Example 1.14 ratio of the density was expressed by equations (1.XIV.l) while here the ratio is expressed by different equations. The difference between the two equations is the fact that Example 1.14 use the integral equation without using any "equation of state." The method described in the Example 1.14 is more general which provided a simple solution[5]. The equation of state suggests that $\partial P = g\,\rho_0\,f(P)\,dz$ while the integral equation is $\Delta P = g\int\rho\,dz$ where no assumption is made on the relationship between the pressure and density. However, the integral equation uses the fact that the pressure is function of location. The comparison between the two methods will be presented.

Example 4.6:

[5]This author is not aware of the "equation of state" solution or the integral solution. If you know of any of these solutions or similar, please pass this information to this author.

## 4.3.4 The Pressure Effects Due To Temperature Variations

### 4.3.4.1 The Basic Analysis

There are situations when the main change of the density results from other effects. For example, when the temperature field is not uniform, the density is affected and thus the pressure is a location function (for example, the temperature in the atmostphere is assumed to be a linear with the height under certain conditions.). A bit more complicate case is when the gas is a function of the pressure and another parameter. Air can be a function of the temperature field and the pressure. For the atmosphere, it is commonly assumed that the temperature is a linear function of the height.

Here, a simple case is examined for which the temperature is a linear function of the height as

$$\frac{dT}{dh} = -C_x \tag{4.48}$$

where $h$ here referred to height or distance. Hence, the temperature–distance function can be written as

$$T = Constant - C_x\, h \tag{4.49}$$

where the $Constant$ is the integration constant which can be obtained by utilizing the initial condition. For $h = 0$, the temperature is $T_0$ and using it leads to

Temp variations

$$\boxed{T = T_0 - C_x\, h} \tag{4.50}$$

Combining equation (4.50) with (4.11) results in

$$\frac{\partial P}{\partial h} = -\frac{g\, P}{R\,(T_0 - C_x\, h)} \tag{4.51}$$

Separating the variables in equation (4.51) and changing the formal $\partial$ to the informal $d$ to obtain

$$\frac{d\,P}{P} = -\frac{g\, dh}{R\,(T_0 - C_x\, h)} \tag{4.52}$$

Defining a new variable[6] as $\xi = (T_0 - C_x\, h)$ for which $\xi_0 = T_0 - C_x\, h_0$ and $d/d\xi = -C_x\, d/dh$. Using these definitions results in

$$\frac{d\,P}{P} = \frac{g}{RC_x}\frac{d\xi}{\xi} \tag{4.53}$$

[6]A colleague asked this author to insert this explanation for his students. If you feel that it is too simple, please, just ignore it.

After the integration of equation (4.52) and reusing (the reverse definitions) the variables transformed the result into

$$\ln \frac{P}{P_0} = \frac{g}{R\,C_x} \ln \frac{T_0 - C_x\,h}{T_0} \tag{4.54}$$

Or in a more convenient form as

Pressure in Atmosphere

$$\frac{P}{P_0} = \left(\frac{T_0 - C_x\,h}{T_0}\right)^{\left(\frac{g}{R\,C_x}\right)} \tag{4.55}$$

It can be noticed that equation (4.55) is a monotonous function which decreases with height because the term in the brackets is less than one. This situation is roughly representing the pressure in the atmosphere and results in a temperature decrease. It can be observed that $C_x$ has a "double role" which can change the pressure ratio. Equation (4.55) can be approximated by two approaches/ideas. The first approximation for a small distance, $h$, and the second approximation for a small temperature gradient. It can be recalled that the following expansions are

$$\frac{P}{P_0} = \lim_{h->0} \left(1 - \frac{C_x}{T_0}\,h\right)^{\frac{g}{R\,C_x}} = 1 - \overbrace{\frac{g\,h}{T_0\,R}}^{\frac{g\,h\,\rho_0}{P_0}} - \overbrace{\frac{\left(R\,g\,C_x - g^2\right)\,h^2}{2\,{T_0}^2\,R^2}}^{\text{correction factor}} - \ldots \tag{4.56}$$

Equation (4.56) shows that the first two terms are the standard terms (negative sign is as expected i.e. negative direction). The correction factor occurs only at the third term which is important for larger heights. It is worth to point out that the above statement has a qualitative meaning when additional parameter is added. However, this kind of analysis will be presented in the dimensional analysis chapter[7].

The second approximation for small $C_x$ is

$$\frac{P}{P_0} = \lim_{C_x->0} \left(1 - \frac{C_x}{T_0}\,h\right)^{\frac{g}{R\,C_x}} = \mathrm{e}^{-\frac{g\,h}{R\,T_0}} - \frac{g\,h^2\,C_x}{2\,{T_0}^2\,R}\,\mathrm{e}^{-\frac{g\,h}{R\,T_0}} - \ldots \tag{4.57}$$

Equation (4.57) shows that the correction factor (lapse coefficient), $C_x$, influences at only large values of height. It has to be noted that these equations (4.56) and (4.57) are not properly represented without the characteristic height. It has to be inserted to make the physical significance clearer.

Equation (4.55) represents only the pressure ratio. For engineering purposes, it is sometimes important to obtain the density ratio. This relationship can be obtained from combining equations (4.55) and (4.50). The simplest assumption to combine these

[7]These concepts are very essential in all the thermo–fluid science. I am grateful to my adviser E.R.G. Eckert who was the pioneer of the dimensional analysis in heat transfer and was kind to show me some of his ideas.

equations is by assuming the ideal gas model, equation (2.25), to yield

$$\frac{\rho}{\rho_0} = \frac{P\,T_0}{P_0\,T} = \overbrace{\left(1 - \frac{C_x\,h}{T_0}\right)^{\left(\frac{g}{R\,C_x}\right)}}^{\frac{P}{P_0}} \overbrace{\left(1 + \frac{C_x\,h}{T}\right)}^{\frac{T_0}{T}} \tag{4.58}$$

*Advance material can be skipped*

#### 4.3.4.2 The Stability Analysis

It is interesting to study whether this solution (4.55) is stable and if so under what conditions. Suppose that for some reason, a small slab of material moves from a layer at height, $h$, to layer at height $h + dh$ (see Figure 4.11) What could happen? There are two main possibilities one: the slab could return to the original layer or two: stay at the new layer (or even move further, higher heights). The first case is referred to as the stable condition and the second case referred to as the unstable condition. The whole system falls apart and does not stay if the analysis predicts unstable conditions. A weak wind or other disturbances can make the unstable system to move to a new condition.

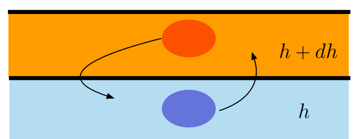


*Fig. -4.11. Two adjoin layers for stability analysis.*

This question is determined by the net forces acting on the slab. Whether these forces are toward the original layer or not. The two forces that act on the slab are the gravity force and the surroundings pressure (buoyant forces). Clearly, the slab is in equilibrium with its surroundings before the movement (not necessarily stable). Under equilibrium, the body forces that acting on the slab are equal to zero. That is, the surroundings "pressure" forces (buoyancy forces) are equal to gravity forces. The buoyancy forces are proportional to the ratio of the density of the slab to surrounding layer density. Thus, the stability question is whether the slab density from layer $h$, $\rho^{'}(h)$ undergoing a free expansion is higher or lower than the density of the layer $h + dh$. If $\rho^{'}(h) > \rho(h+dh)$ then the situation is stable. The term $\rho^{'}(h)$ is slab from layer $h$ that had undergone the free expansion.

The reason that the free expansion is chosen to explain the process that the slab undergoes when it moves from layer $h$ to layer $h + dh$ is because it is the simplest. In reality, the free expansion is not far way from the actual process. The two processes that occurred here are thermal and the change of pressure (at the speed of sound). The thermal process is in the range of [cm/sec] while the speed of sound is about 300 [m/sec]. That is, the pressure process is about thousands times faster than the thermal process. The second issue that occurs during the "expansion" is the shock (in the reverse case $[h + dh] \to h$). However, this shock is insignificant (check book on Fundamentals of Compressible Flow Mechanics by this author on the French problem).

The slab density at layer $h+dh$ can be obtained using equation (4.58) as following

$$\frac{\rho(h+dh)}{\rho(h)} = \frac{P\,T_0}{P_0\,T} = \left(1 - \frac{C_x\,dh}{T_0}\right)^{\left(\frac{g}{R\,C_x}\right)} \left(1 + \frac{C_x\,dh}{T}\right) \tag{4.59}$$

The pressure and temperature change when the slab moves from layer at $h$ to layer $h+dh$. The process, under the above discussion and simplifications, can be assumed to be adiabatic (that is, no significant heat transfer occurs in the short period of time). The little slab undergoes isentropic expansion as following for which (see equation (2.25))

$$\frac{\rho'(h+dh)}{\rho(h)} = \left(\frac{P'(h+dh)}{P(h)}\right)^{1/k} \tag{4.60}$$

When the symbol $'$ denotes the slab that moves from layer $h$ to layer $h+dh$. The pressure ratio is given by equation (4.55) but can be approximated by equation (4.56) and thus

$$\frac{\rho'(h+dh)}{\rho(h)} = \left(1 - \frac{gdh}{T(h)\;R}\right)^{1/k} \tag{4.61}$$

Again using the ideal gas model for equation (4.62) transformed into

$$\frac{\rho'(h+dh)}{\rho(h)} = \left(1 - \frac{\rho\,gdh}{P}\right)^{1/k} \tag{4.62}$$

Expanding equation (4.62) in Taylor series results in

$$\left(1 - \frac{\rho\,gdh}{P}\right)^{1/k} = 1 - \frac{g\,\rho\,dh}{P\,k} - \frac{\left(g^2\,\rho^2\,k - g^2\,\rho^2\right)\,dh^2}{2\,P^2\,k^2} - ... \tag{4.63}$$

The density at layer $h+dh$ can be obtained from (4.59) and then it is expanded in taylor series as

$$\frac{\rho(h+dh)}{\rho(h)} = \left(1 - \frac{C_x\,dh}{T_0}\right)^{\left(\frac{g}{R\,C_x}\right)} \left(1 + \frac{C_x\,dh}{T}\right) \sim 1 - \left(\frac{g\,\rho}{P} - \frac{C_x}{T}\right) dh + \cdots \tag{4.64}$$

The comparison of the right hand terms of equations (4.64) and (4.63) provides the conditions to determine the stability.

From a mathematical point of view, to keep the inequality for a small $dh$ only the first term need to be compared as

$$\frac{g\,\rho}{P\,k} > \frac{g\,\rho}{P} - \frac{C_x}{T} \tag{4.65}$$

After rearrangement of the inequality (4.65) and using the ideal gas identity, it transformed to

$$\frac{C_x}{T} > \frac{(k-1)\,g\,\rho}{k\,P}$$
$$C_x < \frac{k-1}{k}\frac{g}{R} \tag{4.66}$$

The analysis shows that the maximum amount depends on the gravity and gas properties. It should be noted that this value should be changed a bit since the $k$ should be replaced by polytropic expansion $n$. When lapse rate $C_x$ is equal to the right hand side of the inequality, it is said that situation is neutral. However, one has to bear in mind that this analysis only provides a range and isn't exact. Thus, around this value additional analysis is needed [8].

One of the common question this author has been asked is about the forces of continuation. What is the source of the force(s) that make this situation when unstable continue to be unstable? Supposed that the situation became unstable and the layers have been exchanged, would the situation become stable now? One has to remember that temperature gradient forces continuous heat transfer which the source temperature change after the movement to the new layer. Thus, the unstable situation is continuously unstable.

### 4.3.5 Gravity Variations Effects on Pressure and Density

Until now the study focus on the change of density and pressure of the fluid. Equation (4.11) has two terms on the right hand side, the density, $\rho$ and the body force, $g$. The body force was assumed until now to be constant. This assumption must be deviated when the distance from the body source is significantly change. At first glance, the body force is independent of the fluid. The source of the gravity force in gas is another body, while the gravity force source in liquid can be the liquid itself. Thus, the discussion is separated into two different issues. The issues of magnetohydrodynamics are too advance for undergraduate student and therefore,will not be introduced here.

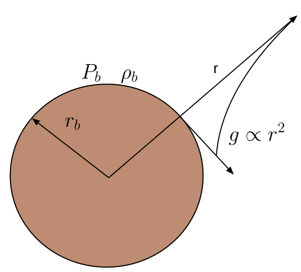


*Fig. -4.12. The varying gravity effects on density and pressure.*

#### 4.3.5.1 Ideal Gas in Varying Gravity

In physics, it was explained that the gravity is a function of the distance from the center of the plant/body. Assuming that the pressure is affected by this gravity/body force. The gravity force is reversely proportional to $r^2$. The gravity force can be assumed that for infinity, $r \to \infty$ the pressure is about zero. Again, equation (4.11) can be used

[8] The same issue of the floating ice. See example for the floating ice in cup.

(semi one directional situation) when $r$ is used as direction and thus

$$\frac{\partial P}{\partial r} = -\rho \frac{G}{r^2} \tag{4.67}$$

where $G$ denotes the general gravity constant. The regular method of separation is employed to obtain

$$\int_{P_b}^{P} \frac{dP}{P} = -\frac{G}{RT} \int_{r_b}^{r} \frac{dr}{r^2} \tag{4.68}$$

where the subscript $b$ denotes the conditions at the body surface. The integration of equation (4.68) results in

$$\ln \frac{P}{P_b} = -\frac{G}{RT} \left( \frac{1}{r_b} - \frac{1}{r} \right) \tag{4.69}$$

Or in a simplified form as

$$\frac{\rho}{\rho_b} = \frac{P}{P_b} = \mathsf{e}^{-\frac{G}{RT}\frac{r-r_b}{r\,r_b}} \tag{4.70}$$

Equation (4.70) demonstrates that the pressure is reduced with the distance. It can be noticed that for $r \rightarrow r_b$ the pressure is approaching $P \rightarrow P_b$. This equation confirms that the density in outer space is zero $\rho(\infty) = 0$. As before, equation (4.70) can be expanded in Taylor series as

$$\frac{\rho}{\rho_b} = \frac{P}{P_b} = \overbrace{1 - \frac{G\,(r-r_b)}{R\,T}}^{standard} - \overbrace{\frac{\left(2\,G\,R\,T + G^2\,r_b\right)\,(r-r_b)^2}{2\,r_b\,(R\,T)^2} + \dots}^{correction\ factor} \tag{4.71}$$

Notice that $G$ isn't our beloved and familiar $g$ and also that $G\,r_b/RT$ is a dimensionless number (later in dimensionless chapter about it and its meaning).

#### 4.3.5.2 Real Gas in Varying Gravity

The regular assumption of constant compressibility, $Z$, is employed. It has to remember when this assumption isn't accurate enough, numerical integration is a possible solution. Thus, equation (4.68) is transformed into

$$\int_{P_b}^{P} \frac{dP}{P} = -\frac{G}{Z\,R\,T} \int_{r_b}^{r} \frac{dr}{r^2} \tag{4.72}$$

With the same process as before for ideal gas case, one can obtain

$$\frac{\rho}{\rho_b} = \frac{P}{P_b} = \mathsf{e}^{-\frac{G}{Z\,R\,T}\frac{r-r_b}{r\,r_b}} \tag{4.73}$$

Equation (4.70) demonstrates that the pressure is reduced with the distance. It can be observed that for $r \rightarrow r_b$ the pressure is approaching $P \rightarrow P_b$. This equation confirms that the density in outer space is zero $\rho(\infty) = 0$. As before Taylor series for equation (4.70) is

$$\frac{\rho}{\rho_b} = \frac{P}{P_b} = \overbrace{1 - \frac{G\,(r - r_b)}{Z\,R\,T}}^{standard} - \overbrace{\frac{\left(2\,G\,Z\,R\,T + G^2\,r_b\right)\,(r - r_b)^2}{2\,r_b\,(Z\,R\,T)^2} + \dots}^{correction\ factor} \tag{4.74}$$

It can be noted that compressibility factor can act as increase or decrease of the ideal gas model depending on whether it is above one or below one. This issue is related to Pushka equation that will be discussed later.

#### 4.3.5.3 Liquid Under Varying Gravity

For comparison reason consider the deepest location in the ocean which is about 11,000 [m]. If the liquid "equation of state" (4.40) is used with the hydrostatic fluid equation results in

$$\frac{\partial P}{\partial r} = -\rho_0\, \mathrm{e}^{\frac{P-P_0}{B_T}} \frac{G}{r^2} \tag{4.75}$$

which the solution of equation (4.75) is

$$\mathrm{e}^{\frac{P_0-P}{B_T}} = Constant - \frac{B_T\,g\,\rho_0}{r} \tag{4.76}$$

Since this author is not aware to which practical situation this solution should be applied, it is left for the reader to apply according to problem, if applicable.

### 4.3.6 Liquid Phase

While for most practical purposes, the Cartesian coordinates provides sufficient treatment to the problem, there are situations where the spherical coordinates must be considered and used.

Derivations of the fluid static in spherical coordinates are

Pressure Spherical Coordinates

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) + 4\,\pi\,G\rho = 0} \tag{4.77}$$

Or in a vector form as

$$\nabla \bullet \left(\frac{1}{\rho}\nabla P\right) + 4\,\pi\,G\rho = 0 \tag{4.78}$$

## 4.4 Fluid in a Accelerated System

Up to this stage, body forces were considered as one-dimensional. In general, the linear acceleration have three components as opposed to the previous case of only one. However, the previous derivations can be easily extended. Equation (4.8) can be transformed into a different coordinate system where the main coordinate is in the direction of the effective gravity. Thus, the previous method can be used and there is no need to solve new three (or two) different equations. As before, the constant pressure plane is perpendicular to the direction of the effective gravity. Generally the acceleration is divided into two categories: linear and angular and they will be discussed in this order.

### 4.4.1 Fluid in a Linearly Accelerated System

For example, in a two dimensional system, for the effective gravity

$$g_{eff} = a\,\hat{i} + g\,\hat{k} \tag{4.79}$$

where the magnitude of the effective gravity is

$$|g_{eff}| = \sqrt{g^2 + a^2} \tag{4.80}$$

and the angle/direction can be obtained from

$$tan\beta = \frac{a}{g} \tag{4.81}$$

Perhaps the best way to explain the linear acceleration is by examples. Consider the following example to illustrate the situation.

Example 4.7:

*A tank filled with liquid is accelerated at a constant acceleration. When the acceleration is changing from the right to the left, what happened to the liquid surface? What is the relative angle of the liquid surface for a container in an accelerated system of $a = 5[m/sec]$?*

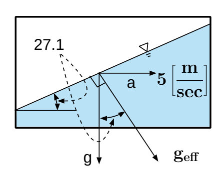



*Fig. -4.13. The effective gravity is for accelerated cart.*

SOLUTION

This question is one of the traditional question of the fluid static and is straight forward. The solution is obtained by finding the effective angle body force. The effective angle is obtained by adding vectors. The change of the acceleration from the right to left is like subtracting vector (addition negative vector). This angle/direction can be found using the following

$$\tan^{-1}\beta = \tan^{-1}\frac{a}{g} = \frac{5}{9.81} \sim 27.01°$$

The magnitude of the effective acceleration is

$$|g_{eff}| = \sqrt{5^2 + 9.81^2} = 11.015[m/sec^2]$$

End Solution

Example 4.8:

*A cart partially filled with liquid and is sliding on an inclined plane as shown in Figure 4.14. Calculate the shape of the surface. If there is a resistance, what will be the angle? What happen when the slope angle is straight (the cart is dropping straight down)?*

SOLUTION

**(a)**

The angle can be found when the acceleration of the cart is found. If there is no resistance, the acceleration in the cart direction is determined from

$$a = \mathbf{g}\sin\beta \tag{4.82}$$

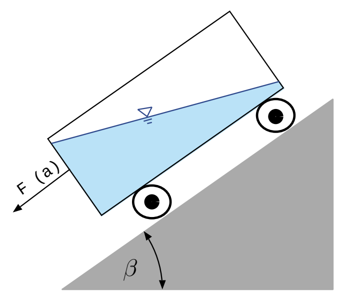



*Fig. -4.14. A cart slide on inclined plane.*

The effective body force is acting perpendicular to the slope. Thus, the liquid surface is parallel to the surface of the inclination surface.

End Solution

**(b)**

In case of resistance force (either of friction due to the air or resistance in the wheels) reduces the acceleration of the cart. In that case the effective body moves closer to the gravity forces. The net body force depends on the mass of the liquid and the net acceleration is

$$a = \mathbf{g} - \frac{F_{net}}{m} \tag{4.83}$$

The angle of the surface, $\alpha < \beta$, is now

$$\tan\alpha = \frac{\mathbf{g} - \frac{F_{net}}{m}}{\mathbf{g}\,cos\beta} \tag{4.84}$$

**(c)**

In the case when the angle of the inclination turned to be straight (direct falling) the effective body force is zero. The pressure is uniform in the tank and no pressure difference can be found. So, the pressure at any point in the liquid is the same and equal to the atmospheric pressure.

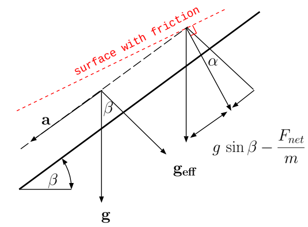


*Fig. -4.15. Forces diagram of cart sliding on inclined plane.*

### 4.4.2 Angular Acceleration Systems: Constant Density

For simplification reasons, the first case deals with a rotation in a perpendicular to the gravity. That effective body force can be written as

$$\mathbf{g}_{eff} = -g\,\hat{k} + \omega^2 r\,\hat{r} \tag{4.85}$$

The lines of constant pressure are not straight lines but lines of parabolic shape. The angle of the line depends on the radius as

$$\frac{dz}{dr} = -\frac{g}{\omega^2 r} \tag{4.86}$$

Equation (4.86) can be integrated as

$$z - z_0 = \frac{\omega^2\,r^2}{2\,g} \tag{4.87}$$

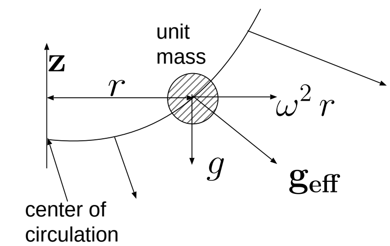


*Fig. -4.16. Schematic to explain the angular angle.*

Notice that the integration constant was substituted by $z_0$. The constant pressure will be along

Angular Acceleration System

$$P - P_0 = \rho g \left[ (z_0 - z) + \frac{\omega^2\,r^2}{2\,g} \right] \tag{4.88}$$

To illustrate this point, example 4.9 is provided.

Example 4.9:

*A "U" tube with a length of $(1+x)L$ is rotating at angular velocity of $\omega$. The center of rotation is a distance, $L$ from the "left" hand side. Because the asymmetrical nature of the problem there is difference in the heights in the U tube arms of $S$ as shown in Figure 4.17. Expresses the relationship between the different parameters of the problem.*

Solution

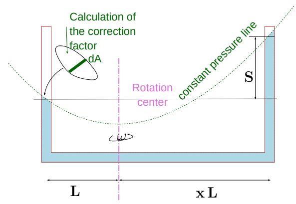



*Fig. -4.17. Schematic angular angle to explain example 4.9.*

It is first assumed the height is uniform at the tube (see for the open question on this assumption). The pressure at the interface at the two sides of the tube is same. Thus, equation (4.87) represent the pressure line. Taking the "left" wing of U tube

$$\overbrace{z_l - z_0}^{\text{change in z direction}} = \overbrace{\frac{\omega^2 L^2}{2\,g}}^{\text{change in r direction}}$$

The same can be said for the other side

$$z_r - z_0 = \frac{\omega^2\, x^2\, L^2}{2\,g}$$

Thus subtracting the two equations above from each each other results in

$$z_r - z_l = \frac{L\,\omega^2\,\left(1 - x^2\right)}{2\,g}$$

It can be noticed that this kind equipment can be used to find the gravity.

End Solution

Example 4.10:
*Assume the diameter of the U tube is $R_t$. What will be the correction factor if the curvature in the liquid in the tube is taken in to account. How would you suggest to define the height in the tube?*

SOLUTION

In Figure 4.17 shows the infinitesimal area used in these calculations. The distance of the infinitesimal area from the rotation center is ?. The height of the infinitesimal area is ?. Notice that the curvature in the two sides are different from each other. The volume above the lower point is ? which is only a function of the geometry.

End Solution

Example 4.11:
*In the U tube in example 4.9 is rotating with upper part height of $\ell$. At what rotating*

*velocity liquid start to exit the U tube? If the rotation of U tube is exactly at the center, what happen the rotation approach very large value?*

*Advance material can be skipped*

### 4.4.3 Fluid Statics in Geological System

*This author would like to express his gratitude to Ralph Menikoff for suggesting this topic.*

In geological systems such as the Earth provide cases to be used for fluid static for estimating pressure. It is common in geology to assume that the Earth is made of several layers. If this assumption is accepted, these layers assumption will be used to do some estimates. The assumption states that the Earth is made from the following layers: solid inner core, outer core, and two layers in the liquid phase with a thin crust. For the purpose of this book, the interest is the calculate the pressure at bottom of the liquid phase. This explaination is provided to understand how to use the bulk

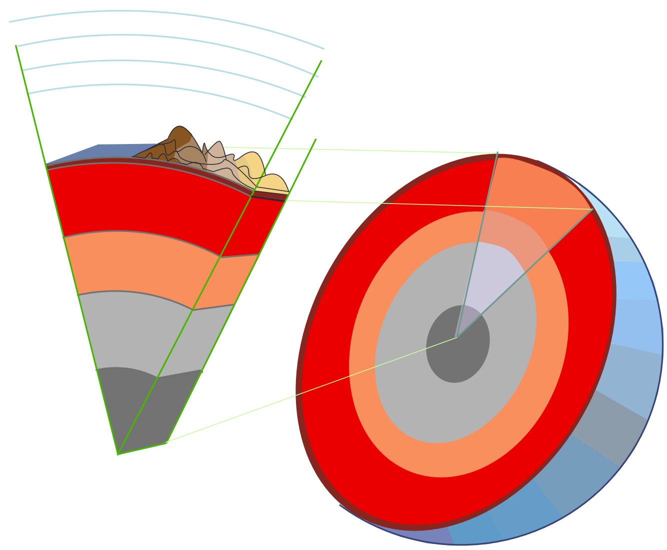


*Fig. -4.18. Earth layers not to scale.*[9]

modulus and the effect of rotation. In reality, there might be an additional effects which affecting the situation but these effects are not the concern of this discussion.

Two different extremes can recognized in fluids between the outer core to the crust. In one extreme is the equator which the rotation play the most significant role.

[9]The image was drawn by Shoshana Bar-Meir, inspired from image made by user Surachit

In the other extreme north–south does not play any effect since the radius is relatively very small. In that case, the pressure at the bottom of the liquid layer can be estimated using the equation (4.45) or in approximation of equation (1.XIV.j). In this case it also can be noticed that $g$ is a function of $r$. If the bulk modulus is assumed constant (for simplicity) governing equation can be constructed starting with equation (1.28). The approximate definition of the bulk modulus is

$$B_T = \frac{\rho\,\Delta P}{\Delta\rho} \Longrightarrow \Delta\rho = \frac{\rho\,\Delta P}{B_T} \tag{4.89}$$

Using equation to express the pressure difference (see Example 1.14 for details explanation) as

$$\rho(r) = \frac{\rho_0}{1 - \displaystyle\int_{R_0}^{r} \frac{g(r)\rho(r)}{B_T(r)}dr} \tag{4.90}$$

In equation (4.90) it is assumed that $B_T$ is a function of pressure and the pressure is a function of the location. Thus, the bulk modulus can be written as a function of the radius, $r$. Again, for simplicity the bulk modulus is assumed to be constant. Hence,

$$\rho(r) = \frac{\rho_0}{1 - \dfrac{1}{B_T}\displaystyle\int_{R_0}^{r} g(r)\rho(r)dr} \tag{4.91}$$

The governing equation can be written using the famous relation for the gravity as

$$\frac{\rho_0}{\rho(r)} = 1 - \frac{1}{B_T}\int_{R_0}^{r} \frac{G}{r^2}\rho(r)dr \tag{4.92}$$

Equation (4.92) is a relatively simple (Fredholm) integral equation. The solution of this equation obtained by differentiation as

$$\frac{\rho_0}{\rho^2}\frac{d\,\rho}{dr} + \frac{G}{r^2}\rho = 0 \tag{4.93}$$

Under variables separation the equation changes to

$$\int_{\rho_0}^{\rho} \frac{\rho_0}{\rho^3}\,d\,\rho = -\int_{R_0}^{r} \frac{G\,dr}{r^2} \tag{4.94}$$

The solution of equation (4.94) is

$$\frac{\rho_0}{2}\left(\frac{1}{{\rho_0}^2} - \frac{1}{\rho^2}\right) = G\left(\frac{1}{R_0} - \frac{1}{r}\right) \tag{4.95}$$

or

$$\rho = \sqrt{\frac{1}{\left(\dfrac{1}{{\rho_0}^2} - \dfrac{2\,G}{\rho_0}\left(\dfrac{1}{R_0} - \dfrac{1}{r}\right)\right)}} \tag{4.96}$$

These equations (4.95) and (4.96) referred to as expanded Pushka equation. The pressure can be calculated since the density is found as

$$\Delta P = \frac{G}{B_T} \int_{R_0}^{r} \frac{1}{\left( \frac{1}{{\rho_0}^2} - \frac{2\,G}{\rho_0} \left( \frac{1}{R_0} - \frac{1}{r} \right) \right)} \frac{dr}{r^2} \tag{4.97}$$

The integral can evaluated numerically or analytically as

$$\Delta P = -\frac{\rho_0 \log \left( \frac{(2\,\rho_0\,G + r)\ R_0 - 2\,r\,\rho_0\,G}{r\,{\rho_0}^2\,R_0} \right)}{2\,G} - \frac{\rho_0 \log (\rho_0)}{G} \tag{4.98}$$

The other issue that related to this topic is, What is the pressure at the equator when the rotation is taken into account. The rotation affects the density since the pressure changes. Thus, mathematical complications caused by the coupling creates additionally difficulty. The integral in equation (4.92) has to include the rotation effects. It can be noticed that the rotation acts in the opposite direction to the gravity. The pressure difference is

$$\Delta P = \int_{R_0}^{r} \rho \left( \frac{G}{r^2} - \omega\, r^2 \right) dr \tag{4.99}$$

Thus the approximated density ratio can be written as

$$\frac{\rho_0}{\rho} = 1 - \frac{1}{B_T} \int_{R_0}^{r} \rho \left( \frac{G}{r^2} - \omega\, r^2 \right) dr \tag{4.100}$$

Taking derivative of the two sides results in

$$-\frac{\rho_0}{\rho^3} = \frac{1}{B_T} \left( \frac{G}{r^2} - \omega\, r^2 \right) dr = 0 \tag{4.101}$$

Integrating equation (4.101)

$$\frac{\rho_0}{2\,\rho^2} = \frac{1}{B_T} \left( \frac{-G}{r} - \frac{\omega\, r^3}{3} \right) \tag{4.102}$$

Where the pressure is obtained by integration as previously was done. The conclusion is that the pressure at the "equator" is substantially lower than the pressure in the north or the south "poles" of the solid core. The pressure difference is due to the large radius. In the range between the two extreme, the effect of rotation is reduced because the radius is reduced. In real liquid, the flow is much more complicated because it is not stationary but have cells in which the liquid flows around. Nevertheless, this analysis gives some indication on the pressure and density in the core.

*End Advance material*

# 4.5 Fluid Forces on Surfaces

The forces that fluids (at static conditions) extracts on surfaces are very important for engineering purposes. This section deals with these calculations. These calculations are divided into two categories, straight surfaces and curved surfaces.

## 4.5.1 Fluid Forces on Straight Surfaces

A motivation is needed before going through the routine of derivations. Initially, a simple case will be examined. Later, how the calculations can be simplified will be shown.

Example 4.12:

*Consider a rectangular shape gate as shown in Figure 4.19. Calculate the minimum forces, $F_1$ and $F_2$ to maintain the gate in position. Assuming that the atmospheric pressure can be ignored.*

SOLUTION

The forces can be calculated by looking at the moment around point "O." The element of moment is $a\,d\xi$ for the width of the gate and is

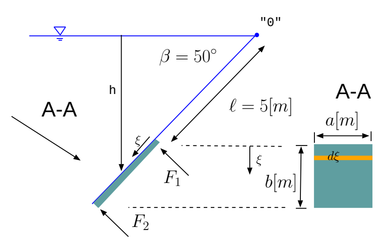



Fig. -4.19. Rectangular area under pressure.

$$dM = \overbrace{P\;\underbrace{a\,d\xi}_{dA}}^{dF}(\ell+\xi)$$

The pressure, $P$ can be expressed as a function $\xi$ as the following

$$P = g\,\rho\,(\ell+\xi) sin\beta$$

The liquid total moment on the gate is

$$M = \int_0^b g\,\rho\,(\ell+\xi)\sin\beta\,a\,d\xi(\ell+\xi)$$

The integral can be simplified as

$$M = g\,a\,\rho\,\sin\beta\int_0^b(\ell+\xi)^2 d\xi \tag{4.103}$$

The solution of the above integral is

$$M = g\,\rho\,a\,\sin\beta\left(\frac{3\,b\,l^2+3\,b^2\,l+b^3}{3}\right)$$

This value provides the moment that $F_1$ and $F_2$ should extract. Additional equation is needed. It is the total force, which is

$$F_{total} = \int_0^b g\,\rho\,(\ell + \xi)\sin\beta\,a\,d\xi$$

The total force integration provides

$$F_{total} = g\,\rho\,a\,\sin\beta \int_0^b (\ell+\xi)d\xi = g\,\rho\,a\,\sin\beta\left(\frac{2\,b\,\ell + b^2}{2}\right)$$

The forces on the gate have to provide

$$F_1 + F_2 = g\,\rho\,a\,\sin\beta\left(\frac{2\,b\,\ell + b^2}{2}\right)$$

Additionally, the moment of forces around point "O" is

$$F_1\,\ell + F_2(\ell + b) = g\,\rho\,a\,\sin\beta\left(\frac{3\,b\,l^2 + 3\,b^2\,l + b^3}{3}\right)$$

The solution of these equations is

$$F_1 = \frac{(3\,\ell + b)\;a\,b\,g\,\rho\,\sin\beta}{6}$$

$$F_2 = \frac{(3\,\ell + 2\,b)\;a\,b\,g\,\rho\,\sin\beta}{6}$$

End Solution

The above calculations are time consuming and engineers always try to make life simpler. Looking at the above calculations, it can be observed that there is a moment of area in equation (4.103) and also a center of area. These concepts have been introduced in Chapter 3. Several represented areas for which moment of inertia and center of area have been tabulated in Chapter 3. These tabulated values can be used to solve this kind of problems.

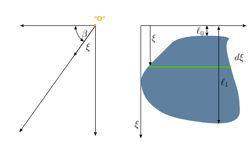


*Fig. -4.20. Schematic of submerged area to explain the center forces and moments.*

### Symmetrical Shapes

Consider the two–dimensional symmetrical area that are under pressure as shown in Figure 4.20. The symmetry is around any axes parallel to axis $x$. The total force and moment that the liquid extracting on the area need to be calculated. First, the force is

$$F = \int_A P dA = \int (P_{atmos} + \rho g h) dA = A\,P_{atmos} + \rho g \int_{\ell_0}^{\ell_1} \overbrace{(\xi + \ell_0)\sin\beta}^{h(\xi)}\,dA \tag{4.104}$$

In this case, the atmospheric pressure can include any additional liquid layer above layer "touching" area. The "atmospheric" pressure can be set to zero.

The boundaries of the integral of equation (4.104) refer to starting point and ending points not to the start area and end area. The integral in equation (4.104) can be further developed as

$$F_{total} = A\,P_{atmos} + \rho\,g\,\sin\beta\left(\ell_0\,A + \overbrace{\int_{\ell_0}^{\ell_1}\xi dA}^{x_c\,A}\right) \tag{4.105}$$

In a final form as

Total Force in Inclined Surface

$$F_{total} = A\left[P_{atmos} + \rho\,g\,\sin\beta\,(\ell_0 + x_c)\right] \tag{4.106}$$

The moment of the liquid on the area around point "O" is

$$M_y = \int_{\xi_0}^{\xi_1} P(\xi)\xi dA \tag{4.107}$$

$$M_y = \int_{\xi_0}^{\xi_1}(P_{atmos} + g\,\rho\,\overbrace{h(\xi)}^{\xi\,\sin\beta})\xi dA \tag{4.108}$$

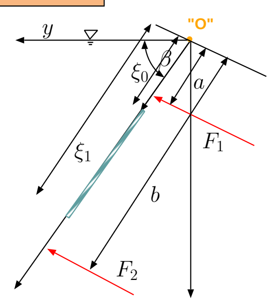



Fig. -4.21. The general forces acting on submerged area.

Or separating the parts as

$$M_y = P_{atmos}\overbrace{\int_{\xi_0}^{\xi_1}\xi dA}^{x_c\,A} + g\,\rho\,\sin\beta\overbrace{\int_{\xi_0}^{\xi_1}\xi^2 dA}^{I_{x'x'}} \tag{4.109}$$

The moment of inertia, $I_{x'x'}$, is about the axis through point "O" into the page. Equation (4.109) can be written in more compact form as

Total Moment in Inclined Surface

$$M_y = P_{atmos}\,x_c\,A + g\,\rho\,\sin\beta I_{x'x'} \tag{4.110}$$

Example 4.12 can be generalized to solve any two forces needed to balance the area/gate. Consider the general symmetrical body shown in figure 4.21 which has two forces that balance the body. Equations (4.106) and (4.110) can be combined the moment and force acting on the general area. If the "atmospheric pressure" can be zero or include additional layer of liquid. The forces balance reads

$$F_1 + F_2 = A\,\left[P_{atmos} + \rho\, g\, \sin\beta\, \left(\ell_0 + x_c\right)\right] \tag{4.111}$$

and moments balance reads

$$F_1\, a + F_2\, b = P_{atmos}\, x_c\, A + g\,\rho\, \sin\beta I_{x'x'} \tag{4.112}$$

The solution of these equations is

$$F1 = \frac{\left[\left(\rho\, \sin\beta - \frac{P_{atmos}}{g\, b}\right)\, x_c + \ell_0\, \rho\, \sin\beta + \frac{P_{atmos}}{g}\right]\, b\, A -, I_{x'x'}\, \rho\, \sin\beta}{g\,(b-a)} \tag{4.113}$$

and

$$F2 = \frac{I_{x'x'}\, \rho\, \sin\beta - \left[\left(\rho\, \sin\beta - \frac{P_{atmos}}{g\, a}\right)\, x_c + \ell_0\, \rho\, \sin\beta + \frac{P_{atmos}}{g}\right]\, a\, A}{g\,(b-a)} \tag{4.114}$$

In the solution, the forces can be negative or positive, and the distance $a$ or $b$ can be positive or negative. Additionally, the atmospheric pressure can contain either an additional liquid layer above the "touching" area or even atmospheric pressure simply can be set up to zero. In symmetrical area only two forces are required since the moment is one dimensional. However, in non–symmetrical area there are two different moments and therefor three forces are required. Thus, additional equation is required. This equation is for the additional moment around the $x$ axis (see for explanation in Figure 4.22). The moment around the $y$ axis is given by equation (4.110) and the total force is given by (4.106). The moment around the $x$ axis (which was arbitrary chosen) should be

$$M_x = \int_A y\, P dA \tag{4.115}$$

Substituting the components for the pressure transforms equation (4.115) into

$$M_x = \int_A y\, \left(P_{atmos} + \rho\, g\, \xi\, \sin\beta\right) dA \tag{4.116}$$

The integral in equation (4.115) can be written as

$$M_x = P_{atmos}\, \overbrace{\int_A y\, dA}^{A\, y_c} + \rho\, g\, \sin\beta\, \overbrace{\int_A \xi\, y\, dA}^{I_{x'y'}} \tag{4.117}$$

The compact form can be written as

Moment in Inclined Surface

$$\boxed{M_x = P_{atmos}\, A\, y_c + \rho\, g\, \sin\beta\, I_{x'y'}} \tag{4.118}$$

The product of inertia was presented in Chapter 3. These equations (4.106), (4.110) and (4.118) provide the base for solving any problem for straight area under pressure with uniform density. There are many combinations of problems (e.g. two forces and moment) but no general solution is provided. Example to illustrate the use of these equations is provided.

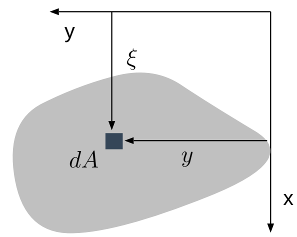



*Fig. -4.22. The general forces acting on non symmetrical straight area.*

Example 4.13:
*Calculate the forces which required to balance the triangular shape shown in the Figure 4.23.*

SOLUTION

The three equations that needs to be solved are

$$F_1 + F_2 + F_3 = F_{total} \tag{4.119}$$

The moment around $x$ axis is

$$F_1\, b = M_y \tag{4.120}$$

The moment around $y$ axis is

$$F_1\, \ell_1 + F_2\,(a + \ell_0) + F_3\, \ell_0 = M_x \tag{4.121}$$

The right hand side of these equations are given before in equations (4.106), (4.110) and (4.118).

The moment of inertia of the triangle around $x$ is made of two triangles (as shown in the Figure (4.23) for triangle 1 and 2). Triangle 1 can be calculated as the moment of inertia around its center which is $\ell_0 + 2*(\ell_1 - \ell_0)/3$. The height of triangle 1 is $(\ell_1 - \ell_0)$ and its width $b$ and thus, moment of inertia about its center is $I_{xx} = b(\ell_1 - \ell_0)^3/36$. The moment of inertia for triangle 1 about $y$ is

$$I_{xx1} = \frac{b(\ell_1-\ell_0)^3}{36} + \overbrace{\frac{b(\ell_1-\ell_0)}{3}}^{A_1} \overbrace{\left(\ell_0 + \frac{2(\ell_1-\ell_0)}{3}\right)^2}^{\Delta {x_1}^2}$$

The height of the triangle 2 is $a - (\ell_1 - \ell_0)$ and its width $b$ and thus, the moment of inertia about its center is

$$I_{xx2} = \frac{b[a-(\ell_1-\ell_0)]^3}{36} + \overbrace{\frac{b[a-(\ell_1-\ell_0)]}{3}}^{A_2} \overbrace{\left(\ell_1 + \frac{[a-(\ell_1-\ell_0)]}{3}\right)^2}^{\Delta {x_2}^2}$$

and the total moment of inertia

$$I_{xx} = I_{xx1} + I_{xx2}$$

The product of inertia of the triangle can be obtain by integration. It can be noticed that upper line of the triangle is $y = \frac{(\ell_1 - \ell_0)x}{b} + \ell_0$. The lower line of the triangle is $y = \frac{(\ell_1 - \ell_0 - a)x}{b} + \ell_0 + a$.

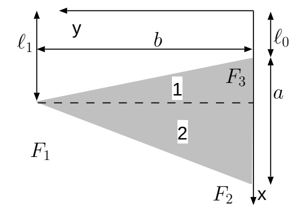



*Fig. -4.23. The general forces acting on a non symmetrical straight area.*

$$I_{xy} = \int_0^b \left[ \int_{\frac{(\ell_1 - \ell_0)x}{b} + \ell_0}^{\frac{(\ell_1 - \ell_0 - a)x}{b} + \ell_0 + a} x\, y\, dx \right] dy = \frac{2\,a\,b^2\,\ell_1 + 2\,a\,b^2\,\ell_0 + a^2\,b^2}{24}$$

The solution of this set equations is

$$F_1 = \overbrace{\left[\frac{a\,b}{3}\right]}^{A} \frac{\left(g\,\left(6\,\ell_1 + 3\,a\right) + 6\,g\,\ell_0\right)\,\rho\,\sin\beta + 8\,P_{atmos}}{24},$$

$$\frac{F_2}{\left[\frac{a\,b}{3}\right]} = -\frac{\left((3\,\ell_1 - 14\,a) - \ell_0\left(\frac{12\,\ell_1}{a} - 27\right) + \frac{12\,{\ell_0}^2}{a}\right) g\,\rho\,\sin\beta}{72} - \frac{\left(\left(\frac{24\,\ell_1}{a} - 24\right) + \frac{48\,\ell_0}{a}\right) P_{atmos}}{72},$$

$$\frac{F_3}{\left[\frac{a\,b}{3}\right]} = \frac{\left(\left(a - \frac{15\,\ell_1}{a}\right) + \ell_0\left(27 - \frac{12\,\ell_1}{a}\right) + \frac{12\,{\ell_0}^2}{a}\right) g\,\rho\,\sin\beta}{72} + \frac{\left(\left(\frac{24\,\ell_1}{a} + 24\right) + \frac{48\,\ell_0}{a}\right) P_{atmos}}{72}$$

End Solution

#### 4.5.1.1 Pressure Center

In the literature, pressure centers are commonly defined. These definitions are mathematical in nature and has physical meaning of equivalent force that will act through this center. The definition is derived or obtained from equation (4.110) and equation (4.118). The pressure center is the distance that will create the moment with the hydrostatic force on point "O." Thus, the pressure center in the $x$ direction is

$$x_p = \frac{1}{F}\int_A x\,P\,dA \tag{4.122}$$

In the same way, the pressure center in the $y$ direction is defined as

$$y_p = \frac{1}{F}\int_A y\,P\,dA \tag{4.123}$$

To show relationship between the pressure center and the other properties, it can be found by setting the atmospheric pressure and $\ell_0$ to zero as following

$$x_p = \frac{g\,\rho \sin\beta\, I_{x'x'}}{A\,\rho\,g\,\sin\beta\,x_c} \tag{4.124}$$

Expanding $I_{x'x'}$ according to equation (3.17) results in

$$x_p = \frac{I_{xx}}{x_c\,A} + x_c \tag{4.125}$$

and in the same fashion in $y$ direction

$$y_p = \frac{I_{xy}}{y_c\,A} + y_c \tag{4.126}$$

It has to emphasis that these definitions are useful only for case where the atmospheric pressure can be neglected or canceled and where $\ell_0$ is zero. Thus, these limitations diminish the usefulness of pressure center definitions. In fact, the reader can find that direct calculations can sometimes simplify the problem.

#### 4.5.1.2 Multiply Layers

In the previous sections, the density was assumed to be constant. For non constant density the derivations aren't "clean" but are similar. Consider straight/flat body that is under liquid with a varying density[10]. If density can be represented by average density, the force that is acting on the body is

$$Geogological F_{total} = \int_A g\,\rho\,h\,dA \sim \bar{\rho} \int_A g\,h\,dA \tag{4.127}$$

In cases where average density cannot be represented reasonably[11], the integral has be carried out. In cases where density is non–continuous, but constant in segments, the following can be said

$$F_{total} = \int_A g\,\rho\,h\,dA = \int_{A_1} g\,\rho_1\,h\,dA + \int_{A_2} g\,\rho_2\,h\,dA + \cdots + \int_{A_n} g\,\rho_n\,h\,dA \tag{4.128}$$

As before for single density, the following can be written

$$F_{total} = g\,\sin\beta \left[ \rho_1 \overbrace{\int_{A_1} \xi\,dA}^{x_{c1}\,A_1} + \rho_2 \overbrace{\int_{A_2} \xi\,dA}^{x_{c2}\,A_2} + \cdots + \rho_n \overbrace{\int_{A_n} \xi\,dA}^{x_{cn}\,A_n} \right] \tag{4.129}$$

[10]This statement also means that density is a monotonous function. Why? Because of the buoyancy issue. It also means that the density can be a non-continuous function.

[11]A qualitative discussion on what is reasonably is not presented here, However, if the variation of the density is within 10% and/or the accuracy of the calculation is minimal, the reasonable average can be used.

Or in a compact form and in addition considering the "atmospheric" pressure can be written as

Total Static Force

$$F_{total} = P_{atmos}\, A_{total} + g\,\sin\beta \sum_{i=1}^{n} \rho_i\, x_{ci}\, A_i \tag{4.130}$$

where the density, $\rho_i$ is the density of the layer $i$ and $A_i$ and $x_{ci}$ are geometrical properties of the area which is in contact with that layer. The atmospheric pressure can be entered into the calculation in the same way as before. Moreover, the atmospheric pressure can include all the layer(s) that do(es) not with the "contact" area.

The moment around axis $y$, $M_y$ under the same considerations as before is

$$M_y = \int_A g\,\rho\,\xi^2\;\sin\beta\,dA \tag{4.131}$$

After similar separation of the total integral, one can find that

$$M_y = g\,\sin\beta \sum_{i=1}^{n} \rho_i\, I_{x'x'\,i} \tag{4.132}$$

If the atmospheric pressure enters into the calculations one can find that

Total Static Moment

$$M_y = P_{atmos}\, x_c\, A_{total} + g\,\sin\beta \sum_{i=1}^{n} \rho_i\, I_{x'x'\,i} \tag{4.133}$$

In the same fashion one can obtain the moment for $x$ axis as

Total Static Moment

$$M_x = P_{atmos}\, y_c\, A_{total} + g\,\sin\beta \sum_{i=1}^{n} \rho_i\, I_{x'y'\,i} \tag{4.134}$$

To illustrate how to work with these equations the following example is provided.

### Example 4.14:

*Consider the hypothetical Figure 4.24. The last layer is made of water with density of* $1000[kg/m^3]$*. The densities are* $\rho_1 = 500[kg/m^3]$*,* $\rho_2 = 800[kg/m^3]$*,* $\rho_3 = 850[kg/m^3]$*, and* $\rho_4 = 1000[kg/m^3]$*. Calculate the forces at points* $a_1$ *and* $b_1$*. Assume that the layers are stables without any movement between the liquids. Also neglect all mass transfer phenomena that may occur. The heights are:* $h_1 = 1[m]$*,* $h_2 = 2[m]$*,* $h_3 = 3[m]$*,and* $h_4 = 4[m]$*. The forces distances are* $a_1 = 1.5[m]$*,* $a_2 = 1.75[m]$*, and* $b_1 = 4.5[m]$*. The angle of inclination is is* $\beta = 45^\circ$*.*

SOLUTION

Since there are only two unknowns, only two equations are needed, which are (4.133) and (4.130). The solution method of this example is applied for cases with less layers (for example by setting the specific height difference to be zero). Equation (4.133) can be used by modifying it, as it can be noticed that instead of using the regular atmospheric pressure the new "atmospheric" pressure can be used as

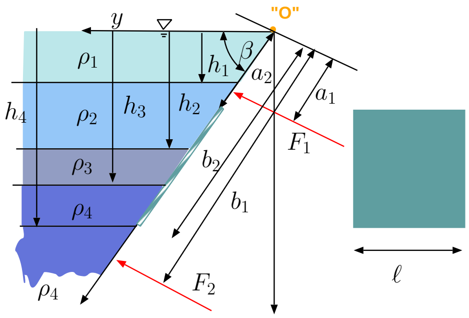



*Fig. -4.24. The effects of multi layers density on static forces.*

$$P_{atmos}{}' = P_{atmos} + \rho_1\, g\, h_1$$

The distance for the center for each area is at the middle of each of the "small" rectangular. The geometries of each areas are

$$
\begin{array}{lll}
x_{c1} = \dfrac{a_2 + \frac{h_2}{\sin\beta}}{2} & A_1 = \ell\left(\frac{h_2}{\sin\beta} - a_2\right) & I_{x'x'1} = \dfrac{\ell\left(\frac{h_2}{\sin\beta} - a_2\right)^3}{36} + (x_{c1})^2\, A_1 \\
x_{c2} = \frac{h_2 + h_3}{2\,\sin\beta} & A_2 = \frac{\ell}{\sin\beta}\,(h_3 - h_2) & I_{x'x'2} = \frac{\ell(h_3 - h_2)^3}{36\,\sin\beta} + (x_{c2})^2\, A_2 \\
x_{c3} = \frac{h_3 + h_4}{2\,\sin\beta} & A_3 = \frac{\ell}{\sin\beta}\,(h_4 - h_3) & I_{x'x'3} = \frac{\ell(h_4 - h_3)^3}{36\,\sin\beta} + (x_{c3})^2\, A_3
\end{array}
$$

After inserting the values, the following equations are obtained

Thus, the first equation is

$$F_1 + F_2 = P_{atmos}{}'\, \overbrace{\ell(b_2 - a_2)}^{A_{total}} + g\,\sin\beta \sum_{i=1}^{3} \rho_{i+1}\, x_{ci}\, A_i$$

The second equation is (4.133) to be written for the moment around the point "O" as

$$F_1\, a_1 + F_2\, b_1 = P_{atmos}{}'\, \overbrace{\frac{(b_2 + a_2)}{2}\,\ell(b_2 - a_2)}^{x_c A_{total}} + g\,\sin\beta \sum_{i=1}^{3} \rho_{i+1}\, I_{x'x'i}$$

The solution for the above equation is

$$F1 = \begin{array}{c} \frac{2\,b_1\,g\,\sin\beta\,\sum_{i=1}^{3}\rho_{i+1}\,x_{ci}\,A_i - 2\,g\,\sin\beta\,\sum_{i=1}^{3}\rho_{i+1}\,I_{x'x'i}}{2\,b_1 - 2\,a_1} - \\[2ex] \frac{\left(b_2{}^2 - 2\,b_1\,b_2 + 2\,a_2\,b_1 - a_2{}^2\right)\ell\, P_{atmos}}{2\,b_1 - 2\,a_1} \end{array}$$

$$F2 = \begin{array}{c} \frac{2\,g\,\sin\beta\,\sum_{i=1}^{3}\rho_{i+1}\,I_{x'x'_i} - 2\,a_1\,g\,\sin\beta\,\sum_{i=1}^{3}\rho_{i+1}\,x_{c\,i}\,A_i}{2\,b_1 - 2\,a_1} + \\ \frac{\left({b_2}^2 + 2\,a_1\,b_2 + {a_2}^2 - 2\,a_1\,a_2\right)\ell\,P_{atmos}}{2\,b_1 - 2\,a_1} \end{array}$$

The solution provided isn't in the complete long form since it will makes things messy. It is simpler to compute the terms separately. A mini source code for the calculations is provided in the the text source. The intermediate results in SI units ([m], $[m^2]$, $[m^4]$) are:

$$\begin{array}{lll} x_{c1} = 2.2892 & x_{c2} = 3.5355 & x_{c3} = 4.9497 \\ A_1 = 2.696 & A_2 = 3.535 & A_3 = 3.535 \\ I_{x'x'1} = 14.215 & I_{x'x'2} = 44.292 & I_{x'x'3} = 86.718 \end{array}$$

The final answer is

$$F_1 = 304809.79[N]$$

and

$$F_2 = 958923.92[N]$$

End Solution

### 4.5.2 Forces on Curved Surfaces

The pressure is acting on surfaces perpendicular to the direction of the surface (no shear forces assumption). At this stage, the pressure is treated as a scalar function. The element force is

$$d\,\mathbf{F} = -P\,\hat{n}\,\mathbf{dA} \qquad (4.135)$$

Here, the conventional notation is used which is to denote the area, $dA$, outward as positive. The total force on the area will be the integral of the unit force

$$\mathbf{F} = -\int_A P\,\hat{n}\,\mathbf{dA} \qquad (4.136)$$

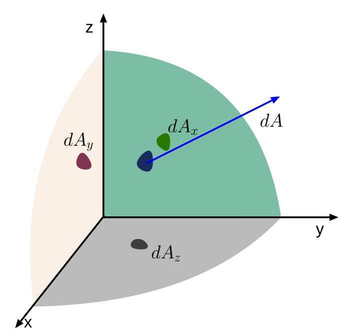



*Fig. -4.25. The forces on curved area.*

The result of the integral is a vector. So, if the $y$ component of the force is needed, only a dot product is needed as

$$dF_y = d\mathbf{F} \bullet \hat{j} \qquad (4.137)$$

From this analysis (equation (4.137)) it can be observed that the force in the direction of $y$, for example, is simply the integral of the area perpendicular to $y$ as

$$F_y = \int_A P \, dA_y \tag{4.138}$$

The same can be said for the $x$ direction.

The force in the $z$ direction is

$$F_z = \int_A h \, g \, \rho dA_z \tag{4.139}$$

The force which acting on the $z$ direction is the weight of the liquid above the projected area plus the atmospheric pressure. This force component can be combined with the other components in the other directions to be

$$F_{total} = \sqrt{{F_z}^2 + {F_x}^2 + {F_y}^2} \tag{4.140}$$

And the angle in "$x\,z$" plane is

$$\tan\theta_{xz} = \frac{F_z}{F_x} \tag{4.141}$$

and the angle in the other plane, "$y\,z$" is

$$\tan\theta_{zy} = \frac{F_z}{F_y} \tag{4.142}$$

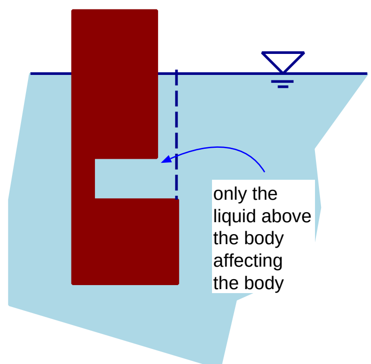



*Fig. -4.26. Schematic of Net Force on floating body.*

The moment due to the curved surface require integration to obtain the value. There are no readily made expressions for these 3–dimensional geometries. However, for some geometries there are readily calculated center of mass and when combined with two other components provide the moment (force with direction line).

### Cut–Out Shapes Effects

There are bodies with a shape that the vertical direction ($z$ direction) is "cut–out" aren't continuous. Equation (4.139) implicitly means that the net force on the body is $z$ direction is only the actual liquid above it. For example, Figure 4.26 shows a floating body with cut–out slot into it. The atmospheric pressure acts on the area with continuous lines. Inside the slot, the atmospheric pressure with it piezometric pressure is canceled by the upper part of the slot. Thus, only the net force is the actual liquid in the slot which is acting on the body. Additional point that is worth mentioning is that the depth where the cut–out occur is insignificant (neglecting the change in the density).

Example 4.15:

*Calculate the force and the moment around point "O" that is acting on the dam (see Figure (4.27)). The dam is made of an arc with the angle of $\theta_0 = 45°$ and radius of $r = 2[m]$. You can assume that the liquid density is constant and equal to 1000 $[kg/m^3]$. The gravity is 9.8$[m/sec^2]$ and width of the dam is $b = 4[m]$. Compare the different methods of computations, direct and indirect.*

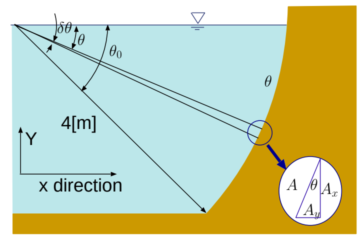



*Fig. -4.27. Calculations of forces on a circular shape dam.*

Solution

The force in the $x$ direction is

$$F_x = \int_A P \overbrace{r\,\cos\theta\,d\theta}^{dA_x} \tag{4.143}$$

Note that the direction of the area is taken into account (sign). The differential area that will be used is, $b\,r\,d\theta$ where $b$ is the width of the dam (into the page). The pressure is only a function of $\theta$ and it is

$$P = P_{atmos} + \rho\,g\,r\sin\theta$$

The force that is acting on the $x$ direction of the dam is $A_x \times P$. When the area $A_x$ is $b\,r\,d\theta\,\cos\theta$. The atmospheric pressure does cancel itself (at least if the atmospheric pressure on both sides of the dam is the same.). The net force will be

$$F_x = \int_0^{\theta_0} \overbrace{\rho\,g\,r\,\sin\theta}^{P} \overbrace{b\,r\,\cos\theta\,d\theta}^{dA_x}$$

The integration results in

$$F_x = \frac{\rho\,g\,b\,r^2}{2}\left(1 - cos^2\left(\theta_0\right)\right)$$

Alternative way to do this calculation is by calculating the pressure at mid point and then multiply it by the projected area, $A_x$ (see Figure 4.28) as

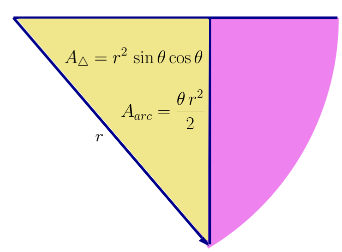



*Fig. -4.28. Area above the dam arc subtract triangle.*

$$F_x = \rho\,g\,\overbrace{b\,r\sin\theta_0}^{A_x}\,\overbrace{\frac{r\sin\theta_0}{2}}^{x_c} = \frac{\rho\,g\,b\,r}{2}\,\sin^2\theta$$

Notice that $dA_x(\cos\theta)$ and $A_x\,(\sin\theta)$ are different, why?

The values to evaluate the last equation are provided in the question and simplify subsidize into it as

$$F_x = \frac{1000 \times 9.8 \times 4 \times 2}{2} \sin(45^\circ) = 19600.0[N]$$

Since the last two equations are identical (use the sinuous theorem to prove it $\sin^2\theta + \cos^2 = 1$), clearly the discussion earlier was right (not a good proof LOL[12]). The force in the $y$ direction is the area times width.

$$F_y = -\overbrace{\left(\overbrace{\frac{\theta_0\, r^2}{2} - \frac{r^2 \sin\theta_0 \cos\theta_0}{2}}^{A}\right)}^{V} b\, g\, \rho \sim 22375.216[N]$$

The center area ( purple area in Figure 4.28) should be calculated as

$$y_c = \frac{y_c\, A_{arc} - y_c\, A_{triangle}}{A}$$

The center area above the dam requires to know the center area of the arc and triangle shapes. Some mathematics are required because the shift in the arc orientation. The arc center (see Figure 4.29) is at

$$y_{c\,arc} = \frac{4\, r \sin^2\left(\frac{\theta}{2}\right)}{3\,\theta}$$

All the other geometrical values are obtained from Tables 3.1 and 3.2. and substituting the proper values results in

$$y_{c_r} = \frac{\overbrace{\frac{\theta\, r^2}{2}}^{A_{arc}} \overbrace{\frac{4\, r \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)}{3\,\theta}}^{y_c} - \overbrace{\frac{2\, r\, \cos\theta}{3}}^{y_c} \overbrace{\frac{\sin\theta\, r^2}{2}}^{A_{triangle}}}{\underbrace{\frac{\theta\, r^2}{2}}_{A_{arc}} - \underbrace{\frac{r^2\, \sin\theta\, \cos\theta}{2}}_{A_{triangle}}}$$

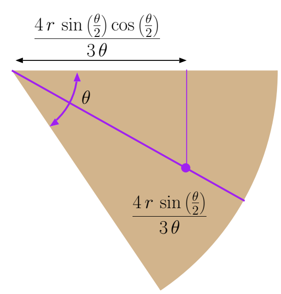



*Fig. -4.29. Area above the dam arc calculation for the center.*

This value is the reverse value and it is

$$y_{c_r} = 1.65174[m]$$

The result of the arc center from point "O" (above calculation area) is

$$y_c = r - y_{c_r} = 2 - 1.65174 \sim 0.348[m]$$

[12]Well, it is just a demonstration!

The moment is

$$M_v = y_c\, F_y \sim 0.348 \times 22375.2 \sim 7792.31759 [N \times m]$$

The center pressure for $x$ area is

$$x_p = x_c + \frac{I_{xx}}{x_c\, A} = \frac{r\, cos\theta_0}{2} + \frac{\overbrace{\frac{\cancel{b}\, (r \cos\theta_0)^3}{36}}^{I_{xx}}}{\underbrace{\frac{r\, cos\theta_0}{2}}_{x_c} \cancel{b}\, (r\ \cos\theta_0)} = \frac{5\, r \cos\theta_0}{9}$$

The moment due to hydrostatic pressure is

$$M_h = x_p\, F_x = \frac{5\, r\, cos\theta_0}{9}\, F_x \sim 15399.21 [N \times m]$$

The total moment is the combination of the two and it is

$$M_{total} = 23191.5 [N \times m]$$

For direct integration of the moment it is done as following

$$dF = P\, dA = \int_0^{\theta_0} \rho\, g\, \sin\theta\, b\, r\, d\theta$$

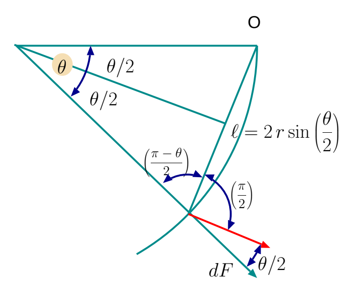



*Fig. -4.30. Moment on arc element around Point "O."*

and element moment is

$$dM = dF \times \ell = dF\, \overbrace{2\, r\, \sin\left(\frac{\theta}{2}\right)}^{\ell} \cos\left(\frac{\theta}{2}\right)$$

and the total moment is

$$M = \int_0^{\theta_0} dM$$

or

$$M = \int_0^{\theta_0} \rho\, g\, \sin\theta\, b\, r\, 2\, r\, \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) d\theta$$

The solution of the last equation is

$$M = \frac{g\, r\, \rho\, (2\, \theta_0 - \sin(2\, \theta_0))}{4}$$

The vertical force can be obtained by

$$F_v = \int_0^{\theta_0} P\, dA_v$$

or

$$F_v = \int_0^{\theta_0} \overbrace{\rho\, g\, r\, \sin\theta}^{P} \overbrace{r\, d\theta\, \cos\theta}^{dA_v}$$

$$F_v = \frac{g\, r^2\, \rho}{2} \left(1 - cos\left(\theta_0\right)^2\right)$$

Here, the traditional approach was presented first, and the direct approach second. It is much simpler now to use the second method. In fact, there are many programs or hand held devices that can carry numerical integration by inserting the function and the boundaries.

End Solution

To demonstrate this point further, consider a more general case of a polynomial function. The reason that a polynomial function was chosen is that almost all the continuous functions can be represented by a Taylor series, and thus, this example provides for practical purposes of the general solution for curved surfaces.

Example 4.16:

*For the liquid shown in Figure 4.31 ,calculate the moment around point "O" and the force created by the liquid per unit depth. The function of the dam shape is $y = \sum_{i=1}^{n} a_i\, x^i$ and it is a monotonous function (this restriction can be relaxed somewhat). Also calculate the horizontal and vertical forces.*

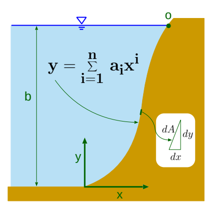



*Fig. -4.31. Polynomial shape dam description for the moment around point "O" and force calculations.*

Solution

The calculations are done per unit depth (into the page) and do not require the actual depth of the dam.

The element force (see Figure 4.31) in this case is

$$dF = \overbrace{\underbrace{(b - y)}_{h}\; g\, \rho}^{P}\; \overbrace{\sqrt{dx^2 + dy^2}}^{dA}$$

The size of the differential area is the square root of the $dx^2$ and $dy^2$ (see Figure 4.31). It can be noticed that the differential area that is used here should be multiplied by the depth. From mathematics, it can be shown that

$$\sqrt{dx^2 + dy^2} = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

The right side can be evaluated for any given function. For example, in this case describing the dam function is

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\sum_{i=1}^{n} i\, a\,(i)\; x\,(i)^{i-1}\right)^2}$$

The value of $x_b$ is where $y = b$ and can be obtained by finding the first and positive root of the equation of

$$0 = \sum_{i=1}^{n} a_i\, x^i - b$$

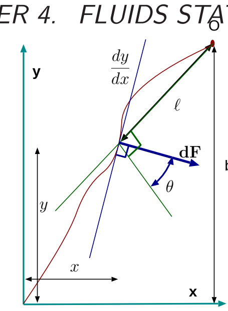



*Fig. -4.32. The difference between the slop and the direction angle.*

To evaluate the moment, expression of the distance and angle to point "O" are needed (see Figure 4.32). The distance between the point on the dam at $x$ to the point "O" is

$$\ell(x) = \sqrt{(b-y)^2 + (x_b - x)^2}$$

The angle between the force and the distance to point "O" is

$$\theta(x) = \tan^{-1}\left(\frac{dy}{dx}\right) - \tan^{-1}\left(\frac{b-y}{x_b - x}\right)$$

The element moment in this case is

$$dM = \ell(x)\; \overbrace{(b-y)\, g\, \rho \sqrt{1+\left(\frac{dy}{dx}\right)^2}}^{dF}\; \cos\theta(x)\, dx$$

To make this example less abstract, consider the specific case of $y = 2\,x^6$. In this case, only one term is provided and $x_b$ can be calculated as following

$$x_b = \sqrt[6]{\frac{b}{2}}$$

Notice that $\sqrt[6]{\frac{b}{2}}$ is measured in meters. The number "2" is a dimensional number with units of $[1/m^5]$. The derivative at $x$ is

$$\frac{dy}{dx} = 12\, x^5$$

and the derivative is dimensionless (a dimensionless number). The distance is

$$\ell = \sqrt{\left(b - 2\,x^6\right)^2 + \left(\sqrt[6]{\frac{b}{2}} - x\right)^2}$$

The angle can be expressed as

$$\theta = \tan^{-1}\left(12\,x^5\right) - \tan^{-1}\left(\frac{b - 2\,x^6}{\sqrt[6]{\frac{b}{2}} - x}\right)$$

The total moment is

$$M = \int_0^{\sqrt[6]{b}} \ell(x)\cos\theta(x)\left(b - 2\,x^6\right)\,g\,\rho\sqrt{1 + 12\,x^5}\,dx$$

This integral doesn't have a analytical solution. However, for a given value $b$ this integral can be evaluate. The horizontal force is

$$F_h = b\,\rho\,g\,\frac{b}{2} = \frac{\rho\,g\,b^2}{2}$$

The vertical force per unit depth is the volume above the dam as

$$F_v = \int_0^{\sqrt[6]{b}} \left(b - 2\,x^6\right)\rho\,g\,dx = \rho\,g\,\frac{5\,b^{\frac{7}{6}}}{7}$$

In going over these calculations, the calculations of the center of the area were not carried out. This omission saves considerable time. In fact, trying to find the center of the area will double the work. This author find this method to be simpler for complicated geometries while the indirect method has advantage for very simple geometries.

End Solution

## 4.6 Buoyancy and Stability

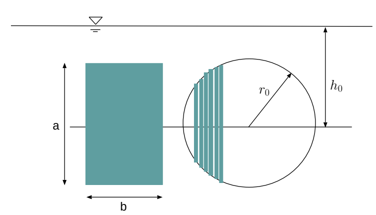


*Fig. -4.33. Schematic of Immersed Cylinder.*

One of the oldest known scientific research on fluid mechanics relates to buoyancy due to question of money was carried by Archimedes. Archimedes principle is related to question of density and volume. While Archimedes did not know much about integrals, he was able to capture the essence. Here, because this material is presented in a different era, more advance mathematics will be used. While the question of the stability was not scientifically examined in the past, the floating vessels structure (more than 150 years ago) show some understanding[13].

The total forces the liquid exacts on a body are considered as a buoyancy issue. To understand this issue, consider a cubical and a cylindrical body that is immersed

[13]This topic was the author's high school name. It was taught by people like these, 150 years ago and more, ship builders who knew how to calculate GM but weren't aware of scientific principles behind it. If the reader wonders why such a class is taught in a high school, perhaps the name can explain it: Sea Officers High School.

in liquid and center in a depth of, $h_0$ as shown in Figure 4.33. The force to hold the cylinder at the place must be made of integration of the pressure around the surface of the square and cylinder bodies. The forces on square geometry body are made only of vertical forces because the two sides cancel each other. However, on the vertical direction, the pressure on the two surfaces are different. On the upper surface the pressure is $\rho g(h_0 - a/2)$. On the lower surface the pressure is $\rho g(h_0 + a/2)$. The force due to the liquid pressure per unit depth (into the page) is

$$F = \rho g \left( (h_0 - a/2) - (h_0 + a/2) \right) \ell \, b = -\rho \, g \, a \, b \, \ell = -\rho g V \tag{4.144}$$

In this case the $\ell$ represents a depth (into the page). Rearranging equation (4.144) to be

$$\frac{F}{V} = \rho \, g \tag{4.145}$$

The force on the immersed body is equal to the weight of the displaced liquid. This analysis can be generalized by noticing two things. All the horizontal forces are canceled. Any body that has a projected area that has two sides, those will cancel each other. Another way to look at this point is by approximation. For any two rectangle bodies, the horizontal forces are canceling each other. Thus even these bodies are in contact with each other, the imaginary pressure make it so that they cancel each other.

On the other hand, any shape is made of many small rectangles. The force on every rectangular shape is made of its weight of the volume. Thus, the total force is made of the sum of all the small rectangles which is the weight of the sum of all volume.

In illustration of this concept, consider the cylindrical shape in Figure 4.33. The force per area (see Figure 4.34) is

$$dF = \overbrace{\rho \, g \, \left( h_0 - r \, \sin\theta \right)}^{P} \overbrace{\sin\theta \, r \, d\theta}^{dA_{vertical}} \tag{4.146}$$

The total force will be the integral of the equation (4.146)

$$F = \int_0^{2\pi} \rho \, g \, \left( h_0 - r \, \sin\theta \right) r \, d\theta \, \sin\theta \tag{4.147}$$

Rearranging equation (4.146) transforms it to

$$F = r \, g \, \rho \int_0^{2\pi} \left( h_0 - r \, \sin\theta \right) \sin\theta \, d\theta \tag{4.148}$$

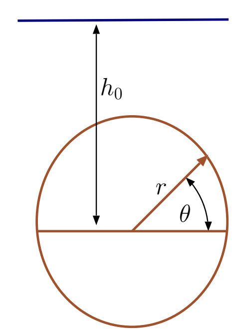



*Fig. -4.34. The floating forces on Immersed Cylinder.*

The solution of equation (4.148) is

$$F = -\pi \, r^2 \, \rho \, g \tag{4.149}$$

The negative sign indicate that the force acting upwards. While the horizontal force is

$$F_v = \int_0^{2\,\pi} \left( h_0 - r \, \sin\theta \right) \cos\theta \, d\theta = 0 \tag{4.150}$$

Example 4.17:

*To what depth will a long log with radius, $r$, a length, $\ell$ and density, $\rho_w$ in liquid with denisty, $\rho_l$. Assume that $\rho_l > \rho_w$. You can provide that the angle or the depth.*

Typical examples to explain the buoyancy are of the vessel with thin walls put upside down into liquid. The second example of the speed of the floating bodies. Since there are no better examples, these examples are a must.

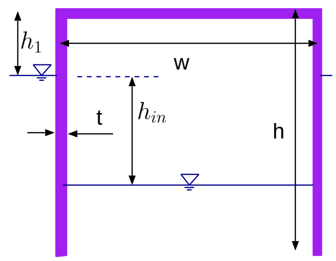



*Fig. -4.35. Schematic of a thin wall floating body.*

Example 4.18:

*A cylindrical body, shown in Figure 4.35 ,is floating in liquid with density, $\rho_l$. The body was inserted into liquid in a such a way that the air had remained in it. Express the maximum wall thickness, $t$, as a function of the density of the wall, $\rho_s$ liquid density, $\rho_l$ and the surroundings air temperature, $T_1$ for the body to float. In the case where thickness is half the maximum, calculate the pressure inside the container. The container diameter is $w$. Assume that the wall thickness is small compared with the other dimensions ($t << w$ and $t << h$).*

<u>Solution</u>

The air mass in the container is

$$m_{air} = \overbrace{\pi\, w^2\, h}^{V} \overbrace{\frac{P_{atmos}}{R T}}^{\rho_{air}}$$

The mass of the container is

$$m_{container} = \left( \overbrace{\pi\, w^2 + 2\, \pi\, w\, h}^{A} \right) t\, \rho_s$$

The liquid amount enters into the cavity is such that the air pressure in the cavity equals to the pressure at the interface (in the cavity). Note that for the maximum thickness, the height, $h_1$ has to be zero. Thus, the pressure at the interface can be written as

$$P_{in} = \rho_l\, g\, h_{in}$$

On the other hand, the pressure at the interface from the air point of view (ideal gas model) should be

$$P_{in} = \frac{m_{air}\, R\, T_1}{\underbrace{h_{in}\, \pi\, w^2}_{V}}$$

Since the air mass didn't change and it is known, it can be inserted into the above equation.

$$\rho_l\, g\, h_{in} + P_{atmos} = P_{in} = \frac{\left(\pi\, w^2\, h\right) \overbrace{\frac{P_{atmos}}{R\,T_1}}^{\rho} R\,T_1}{h_{in}\,\pi\, w^2}$$

The last equation can be simplified into

$$\rho_l\, g\, h_{in} + P_{atmos} = \frac{h\, P_{atmos}}{h_{in}}$$

And the solution for $h_{in}$ is

$$h_{in} = -\frac{P_{atmos} + \sqrt{4\,g\,h\,P_{atmos}\,\rho_l + {P_{atmos}}^2}}{2\,g\,\rho_l}$$

and

$$h_{in} = \frac{\sqrt{4\,g\,h\,P_{atmos}\,\rho_l + {P_{atmos}}^2} - P_{atmos}}{2\,g\,\rho_l}$$

The solution must be positive, so that the last solution is the only physical solution.

Example 4.19:

*Calculate the minimum density an infinitely long equilateral triangle (three equal sides) has to be so that the sharp end is in the water.*

*Advance material can be skipped*

**Extreme Cases**

The solution demonstrates that when $h \longrightarrow 0$ then $h_{in} \longrightarrow 0$. When the gravity approaches zero (macro gravity) then

$$h_{in} = \frac{P_{atmos}}{\rho_l\, g} + h - \frac{h^2\,\rho_l\, g}{P_{atmos}} + \frac{2\,h^3\,{\rho_l}^2\,g^2}{{P_{atmos}}^2} - \frac{5\,h^4\,{\rho_l}^3\,g^3}{{P_{atmos}}^3} + \cdots$$

This "strange" result shows that bodies don't float in the normal sense. When the floating is under vacuum condition, the following height can be expanded into

$$h_{in} = \sqrt{\frac{h\,P_{atmos}}{g\,\rho_l}} + \frac{P_{atmos}}{2\,g\,\rho_l} + \cdots$$

which shows that the large quantity of liquid enters into the container as it is expected.

*End Advance material*

Archimedes theorem states that the force balance is at displaced weight liquid (of the same volume) should be the same as the container, the air. Thus,

$$\overbrace{\pi\, w^2\,(h - h_{in})\, g}^{\text{net displayed water}} = \overbrace{\left(\pi\, w^2 + 2\,\pi\, w\, h\right)\, t\,\rho_s\, g}^{\text{container}} + \overbrace{\pi\, w^2\, h\left(\frac{P_{atmos}}{R\,T_1}\right) g}^{\text{air}}$$

If air mass is neglected the maximum thickness is

$$t_{max} = \frac{2\,g\,h\,w\,\rho_l + P_{atmos}\,w - w\,\sqrt{4\,gh\,P_{atmos}\,\rho_l + {P_{atmos}}^2}}{(2\,g\,w + 4\,g\,h)\;\rho_l\,\rho_s}$$

The condition to have physical value for the maximum thickness is

$$2\,g\,h\,\rho_l + P_{atmos} \geq \sqrt{4\,gh\,P_{atmos}\,\rho_l + {P_{atmos}}^2}$$

The full solution is

$$t_{max} = -\frac{\left(w\,R\,\sqrt{4\,gh\,P_{atmos}\,\rho_l + {P_{atmos}}^2} - 2\,g\,h\,w\,R\,\rho_l - P_{atmos}\,w\,R\right)T_1 + 2\,g\,h\,P_{atmos}\,w\,\rho_l}{(2\,g\,w + 4\,g\,h)\,R\,\rho_l\,\rho_s\,T_1}$$

In this analysis the air temperature in the container immediately after insertion in the liquid has different value from the final temperature. It is reasonable as the first approximation to assume that the process is adiabatic and isentropic. Thus, the temperature in the cavity immediately after the insertion is

$$\frac{T_i}{T_f} = \left(\frac{P_i}{P_f}\right)$$

The final temperature and pressure were calculated previously. The equation of state is

$$P_i = \frac{m_{air}\,R\,T_i}{V_i}$$

The new unknown must provide additional equation which is

$$V_i = \pi\,w^2\,h_i$$

**Thickness Below The Maximum**

For the half thickness $t = \frac{t_{max}}{2}$ the general solution for any given thickness below maximum is presented. The thickness is known, but the liquid displacement is still unknown. The pressure at the interface (after long time) is

$$\rho_l\,g\,h_{in} + P_{atmos} = \frac{\pi\,w^2\,h\,\frac{P_{atmos}}{R\,T_1}\,R\,T_1}{(h_{in} + h_1)\;\pi\,w^2}$$

which can be simplified to

$$\rho_l\,g\,h_{in} + P_{atmos} = \frac{h\,P_{atmos}}{h_{in} + h_1}$$

The second equation is Archimedes' equation, which is

$$\pi\,w^2\,(h - h_{in} - h_1) = \left(\pi\,w^2 + 2\,\pi\,w\,h)\,t\,\rho_s\,g\right) + \pi\,w^2\,h\,\left(\frac{P_{atmos}}{R\,T_1}\right)\,g$$

End Solution

Example 4.20:

*A body is pushed into the liquid to a distance, $h_0$ and left at rest. Calculate acceleration and time for a body to reach the surface. The body's density is $\alpha\,\rho_l$ , where $\alpha$ is ratio between the body density to the liquid density and ($0 < \alpha < 1$). Is the body volume important?*

SOLUTION

The net force is

$$F = \overbrace{V\,g\,\rho_l}^{\text{liquid weight}} - \overbrace{V\,g\,\alpha\,\rho_l}^{\text{body weight}} = V\,g\,\rho_l\,(1-\alpha)$$

But on the other side the internal force is

$$F = m\,a = \overbrace{V\,\alpha\rho_l}^{m}\,a$$

Thus, the acceleration is

$$a = g\left(\frac{1-\alpha}{\alpha}\right)$$

If the object is left at rest (no movement) thus time will be ($h = 1/2\,a\,t^2$)

$$t = \sqrt{\frac{2\,h\alpha}{g(1-\alpha)}}$$

If the object is very light ($\alpha \longrightarrow 0$) then

$$t_{min} = \sqrt{\frac{2\,h\,\alpha}{g}} + \frac{\sqrt{2\,g\,h}\,\alpha^{\frac{3}{2}}}{2\,g} + \frac{3\,\sqrt{2\,g\,h}\,\alpha^{\frac{5}{2}}}{8\,g} + \frac{5\,\sqrt{2\,g\,h}\,\alpha^{\frac{7}{2}}}{16\,g} + \cdots$$

From the above equation, it can be observed that only the density ratio is important. This idea can lead to experiment in "large gravity" because the acceleration can be magnified and it is much more than the reverse of free falling.

End Solution

Example 4.21:

*In some situations, it is desired to find equivalent of force of a certain shape to be replaced by another force of a "standard" shape. Consider the force that acts on a half sphere. Find equivalent cylinder that has the same diameter that has the same force.*

SOLUTION

The force act on the half sphere can be found by integrating the forces around the sphere. The element force is

$$dF = (\rho_L - \rho_S)\, g\, \overbrace{\overbrace{r\ \cos\phi\ \cos\theta}^{h} \cos\theta\ \cos\phi\ \overbrace{r^2\, d\theta\, d\phi}^{dA}}^{dA_x}$$

The total force is then

$$F_x = \int_0^{\pi}\int_0^{\pi} (\rho_L - \rho_S)\, g \cos^2\phi \cos^2\theta\, r^3\, d\theta\, d\phi$$

The result of the integration the force on sphere is

$$F_s = \frac{\pi^2\,(\rho_L - \rho_S)\, r^3}{4}$$

The force on equivalent cylinder is

$$F_c = \pi\, r^2\,(\rho_L - \rho_S)\, h$$

These forces have to be equivalent and thus

$$\frac{\pi^{\cancel{2}}\,\cancel{(\rho_L - \rho_S)}\, r^{\cancelto{1}{3}}}{4} = \cancel{\pi}\cancel{r^2}\,\cancel{(\rho_L - \rho_S)}\, h$$

Thus, the height is

$$\frac{h}{r} = \frac{\pi}{4}$$

End Solution

Example 4.22:

*In the introduction to this section, it was assumed that above liquid is a gas with inconsequential density. Suppose that the above layer is another liquid which has a bit lighter density. Body with density between the two liquids, $\rho_l < \rho_s < rho_h$ is floating between the two liquids. Develop the relationship between the densities of liquids and solid and the location of the solid cubical. There are situations where density is a function of the depth. What will be the location of solid body if the liquid density varied parabolically.*

Solution

In the discussion to this section, it was shown that net force is the body volume times the the density of the liquid. In the same vein, the body can be separated into two: one in first liquid and one in the second liquid. In this case there are two different liquid densities. The net force down is the weight of the body $\rho_c\, h\, A$. Where $h$ is the height of the body and $A$ is its cross section. This force is balance according to above explanation by the two liquid as

$$\rho_c \cancel{h\, A} = \cancel{A\, h}\, (\alpha\, \rho_l + (1-\alpha)\rho_h)$$

Where $\alpha$ is the fraction that is in low liquid. After rearrangement it became

$$\alpha = \frac{\rho_c - \rho_h}{\rho_l - \rho_h}$$

the second part deals with the case where the density varied parabolically. The density as a function of $x$ coordinate along $h$ starting at point $\rho_h$ is

$$\rho(x) = \rho_h - \left(\frac{x}{h}\right)^2 (\rho_h - \rho_l)$$

Thus the equilibration will be achieved, $A$ is canceled on both sides, when

$$\rho_c \, h = \int_{x_1}^{x_1+h} \left[\rho_h - \left(\frac{x}{h}\right)^2 (\rho_h - \rho_l)\right] dx$$

After the integration the equation transferred into

$$\rho_c \, h = \frac{(3\,\rho_l - 3\,\rho_h)\; x1^2 + (3\,h\,\rho_l - 3\,h\,\rho_h)\; x1 + h^2\,\rho_l + 2\,h^2\,\rho_h}{3\,h}$$

And the location where the lower point of the body (the physical), $x_1$, will be at

$$X_1 = \frac{\sqrt{3}\,\sqrt{3\,h^2\,{\rho_l}^2 + (4\,\rho_c - 6\,h^2\,\rho_h)\;\rho_l + 3\,h^2\,{\rho_h}^2 - 12\,\rho_c\,\rho_h} + 3\,h\,\rho_l - 3\,h\,\rho_h}{6\,\rho_h - 2\,\rho_l}$$

For linear relationship the following results can be obtained.

$$x_1 = \frac{h\,\rho_l + h\,\rho_h - 6\,\rho_c}{2\,\rho_l - 2\,\rho_h}$$

In many cases in reality the variations occur in small zone compare to the size of the body. Thus, the calculations can be carried out under the assumption of sharp change. However, if the body is smaller compare to the zone of variation, they have to accounted for.

End Solution

Example 4.23:

*A hollow sphere is made of steel ($\rho_s/\rho_w \cong 7.8$) with a $t$ wall thickness. What is the thickness if the sphere is neutrally buoyant? Assume that the radius of the sphere is $R$. For the thickness below this critical value, develop an equation for the depth of the sphere.*

Solution

The weight of displaced water has to be equal to the weight of the sphere

$$\rho_s \cancel{g} \frac{4\,\pi\,R^3}{3} = \rho_w \cancel{g} \left(\frac{4\,\pi\,R^3}{3} - \frac{4\,\pi\,(R-t)^3}{3}\right) \tag{4.XXIII.a}$$

after simplification equation (4.XXIII.a) becomes

$$\frac{\rho_s\, R^3}{\rho_w} = 3\, t\, R^2 - 3\, t^2\, R + t^3 \tag{4.XXIII.b}$$

Equation (4.XXIII.b) is third order polynomial equation which it's solution (see the appendix) is

$$\begin{aligned} t_1 &= \left(-\frac{\sqrt{3}\, i}{2} - \frac{1}{2}\right)\left(\frac{\rho_s}{\rho_w} R^3 - R^3\right)^{\frac{1}{3}} + R \\ t_2 &= \left(\frac{\sqrt{3}\, i}{2} - \frac{1}{2}\right)\left(\frac{\rho_s}{\rho_w} R^3 - R^3\right)^{\frac{1}{3}} + R \\ t_3 &= R\left(\sqrt[3]{\frac{\rho_s}{\rho_w} - 1} + 1\right) \end{aligned} \tag{4.XXIII.c}$$

The first two solutions are imaginary thus not valid for the physical world. The last solution is the solution that was needed. The depth that sphere will be located depends on the ratio of $t/R$ which similar analysis to the above. For a given ratio of $t/R$, the weight displaced by the sphere has to be same as the sphere weight. The volume of a sphere cap (segment) is given by

$$V_{cap} = \frac{\pi\, h^2\, (3R - h)}{3} \tag{4.XXIII.d}$$

Where $h$ is the sphere height above the water. The volume in the water is

$$V_{water} = \frac{4\,\pi\, R^3}{3} - \frac{\pi\, h^2\, (3R - h)}{3} = \frac{4\,\pi\,\left(R^3 - 3\, R\, h^2 + h^3\right)}{3} \tag{4.XXIII.e}$$

When $V_{water}$ denotes the volume of the sphere in the water. Thus the Archimedes law is

$$\frac{\rho_w\, 4\,\pi\,\left(R^3 - 3\, R\, h^2 + h^3\right)}{3} = \frac{\rho_s\, 4\,\pi\,\left(3\, t\, R^2 - 3\, t^2\, R + t^3\right)}{3} \tag{4.XXIII.f}$$

or

$$\left(R^3 - 3\, R\, h^2 + h^3\right) = \frac{\rho_w}{\rho_s}\left(3\, t\, R^2 - 3\, t^2\, R + t^3\right) \tag{4.XXIII.g}$$

The solution of (4.XXIII.g) is

$$h = \left(\frac{\sqrt{-f R\, \left(4\, R^3 - f R\right)}}{2} - \frac{f R - 2\, R^3}{2}\right)^{\dfrac{1}{3}} + \frac{R^2}{\left(\dfrac{\sqrt{-f R\, \left(4\, R^3 - f R\right)}}{2} - \dfrac{f R - 2\, R^3}{2}\right)^{\dfrac{1}{3}}} \tag{4.XXIII.h}$$

Where $-f R = R^3 - \dfrac{\rho_w}{\rho_s}\left(3\,t\,R^2 - 3\,t^2\,R + t^3\right)$ There are two more solutions which contains the imaginary component. These solutions are rejected.

End Solution

Example 4.24:
*One of the common questions in buoyancy is the weight with variable cross section and fix load. For example, a wood wedge of wood with a fix weight/load. The general question is at what the depth of the object (i.e. wedge) will be located. For simplicity, assume that the body is of a solid material.*

SOLUTION

It is assumed that the volume can be written as a function of the depth. As it was shown in the previous example, the relationship between the depth and the displaced liquid volume of the sphere. Here it is assumed that this relationship can be written as

$$V_w = f(d, \text{other geometrical parameters}) \tag{4.XXIV.a}$$

The Archimedes balance on the body is

$$\rho_\ell V_a = \rho_w V_w \tag{4.XXIV.b}$$

$$d = f^{-1}\frac{\rho_\ell V_a}{\rho_w} \tag{4.XXIV.c}$$

End Solution

Example 4.25:
*In example 4.24 a general solution was provided. Find the reverse function, $f^{-1}$ for cone with $30^\circ$ when the tip is in the bottom.*

SOLUTION

First the function has to built for $d$ (depth).

$$V_w = \frac{\pi\,d\left(\dfrac{d}{\sqrt{3}}\right)^2}{3} = \frac{\pi\,d^3}{9} \tag{4.XXV.a}$$

Thus, the depth is

$$d = \sqrt[3]{\frac{9\,\pi\,\rho_w}{\rho_\ell\,V_a}} \tag{4.XXV.b}$$

End Solution

### 4.6.1 Stability

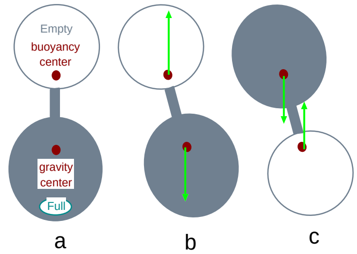



*Fig. -4.36. Schematic of floating bodies.*

Figure 4.36 shows a body made of hollow balloon and a heavy sphere connected by a thin and light rod. This arrangement has mass centroid close to the middle of the sphere. The buoyant center is below the middle of the balloon. If this arrangement is inserted into liquid and will be floating, the balloon will be on the top and sphere on the bottom. Tilting the body with a small angle from its resting position creates a shift in the forces direction (examine Figure 4.36b). These forces create a moment which wants to return the body to the resting (original) position. When the body is at the position shown in Figure 4.36c ,the body is unstable and any tilt from the original position creates moment that will further continue to move the body from its original position. This analysis doesn't violate the second law of thermodynamics. Moving bodies from an unstable position is in essence like a potential.

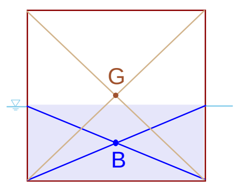



*Fig. -4.37. Schematic of floating cubic.*

A wooden cubic (made of pine, for example) is inserted into water. Part of the block floats above water line. The cubic mass (gravity) centroid is in the middle of the cubic. However the buoyant center is the middle of the volume under the water (see Figure 4.37). This situation is similar to Figure 4.36c. However, any experiment of this cubic wood shows that it is stable locally. Small amount of tilting of the cubic results in returning to the original position. When tilting a larger amount than $\pi/4$ , it results in a flipping into the next stable position. The cubic is stable in six positions (every cubic has six faces). In fact, in any of these six positions, the body is in situation like in 4.36c. The reason for this local stability of the cubic is that other positions are less stable. If one draws the stability (later about this criterion) as a function of the rotation angle will show a sinusoidal function with four picks in a whole rotation.

So, the body stability must be based on the difference between the body's local positions rather than the "absolute" stability. That is, the body is "stable" in some points more than others in their vicinity. These points are raised from the buoyant force analysis. When the body is tilted at a small angle, $\beta$, the immersed part of the body center changes to a new location, **B'** as shown in Figure 4.38. The center of the mass (gravity) is still in the old location since the body did not change. The stability of the body is divided into three categories. If the new immerse volume creates a new center in such way that couple forces (gravity and buoyancy) try to return the body, the original state is referred as the stable body and vice versa. The third state is when the couple forces do have zero moment, it is referred to as the neutral stable.

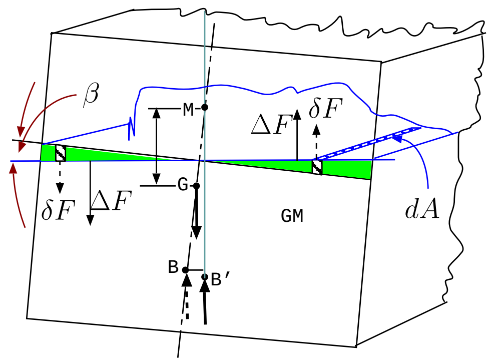



*Fig. -4.38. Stability analysis of floating body.*

The body, shown in Figure 4.38, when given a tilted position, move to a new buoyant center, **B'**. This deviation of the buoyant center from the old buoyant center location, **B**, should to be calculated. This analysis is based on the difference of the displaced liquid. The right green area (volume) in Figure 4.38 is displaced by the same area (really the volume) on left since the weight of the body didn't change[14] so the total immersed section is constant. For small angle, $\beta$, the moment is calculated as the integration of the small force shown in the Figure 4.38 as $\Delta F$. The displacement of the buoyant center can be calculated by examining the moment these forces creats. The body weight creates opposite moment to balance the moment of the displaced liquid volume.

$$\overline{BB'}\,W = \mathbf{M} \tag{4.151}$$

Where $\mathbf{M}$ is the moment created by the displaced areas (volumes), $\overline{BB'}$ is the distance between points **B** and point **B'**, and, $W$ referred to the weight of the body. It can be noticed that the distance $\overline{BB'}$ is an approximation for small angles (neglecting the vertical component.). So the perpendicular distance, $\overline{BB'}$, should be

$$\overline{BB'} = \frac{\mathbf{M}}{W} \tag{4.152}$$

The moment $\mathbf{M}$ can be calculated as

$$\mathbf{M} = \int_A \overbrace{g\,\rho_l\,\underbrace{x\,\beta\,dA}_{dV}}^{\delta F} x = g\,\rho_l\,\beta \int_A x^2 dA \tag{4.153}$$

[14]It is correct to state: area only when the body is symmetrical. However, when the body is not symmetrical, the analysis is still correct because the volume and not the area is used.

The integral in the right side of equation (4.153) is referred to as the area moment of inertia and was discussed in Chapter 3. The distance, $\overline{BB'}$ can be written from equation (4.153) as

$$\overline{BB'} = \frac{g\,\rho_l\,I_{xx}}{\rho_s V_{body}} \tag{4.154}$$

The point where the gravity force direction is intersecting with the center line of the cross section is referred as metacentric point, **M**. The location of the metacentric point can be obtained from the geometry as

$$\overline{BM} = \frac{\overline{BB'}}{\sin\beta} \tag{4.155}$$

And combining equations (4.154) with (4.155) yields

$$\overline{BM} = \frac{\cancel{g}\,\rho_l \beta I_{xx}}{\cancel{g}\,\rho_s\,\sin\beta\,V_{body}} = \frac{\rho_l\,I_{xx}}{\rho_s\,V_{body}} \tag{4.156}$$

For small angle ($\beta \sim 0$)

$$\lim_{\beta\to 0}\frac{\sin\beta}{\beta} \sim 1 \tag{4.157}$$

It is remarkable that the results is independent of the angle. Looking at Figure 4.38, the geometrical quantities can be related as

$$\overline{GM} = \overbrace{\frac{\rho_l\,I_{xx}}{\rho_s V_{body}}}^{\overline{BM}} - \overline{BG} \tag{4.158}$$

Example 4.26:

*A solid cone floats in a heavier liquid (that is $\rho_l/\rho_c > 1$). The ratio of the cone density to liquid density is $\alpha$. For a very light cone $\rho_c/\rho_l \sim 0$, the cone has zero depth. At this condition, the cone is unstable. For middle range, $1 > \rho_c/\rho_l > 0$ there could be a range where the cone is stable. The angle of the cone is $\theta$. Analyze this situation.*

<u>Solution</u>

The floating cone volume is $\dfrac{\pi\,d\,r^2}{3}$ and the center of gravity is D/4. The distance $\overline{BG}$ depent on $d$ as

$$\overline{BG} = D/4 - d/4 \tag{4.XXVI.a}$$

Where $D$ is the total height and $d$ is the height of the submerged cone. The moment of inertia of the cone is circle shown in Table 3.1. The relationship between the radius the depth is

$$r = d\,\tan\theta \tag{4.XXVI.b}$$

$$\overline{GM} = \frac{\rho_l \overbrace{\frac{\pi\,(d\,\tan\theta)^4}{64}}^{I_{xx}}}{\rho_s \underbrace{\frac{\pi\,d\,(d\,\tan\theta)^2}{3}}_{V_{body}}} - \overbrace{\left(\frac{D}{4} - \frac{d}{4}\right)}^{\overline{BG}} \tag{4.XXVI.c}$$

Equation (4.XXVI.c) can be simplified as

$$\overline{GM} = \frac{\rho_l\,d\,\tan^2\theta}{\rho_s\,192} - \left(\frac{D}{4} - \frac{d}{4}\right) \tag{4.XXVI.d}$$

The relationship between $D$ and $d$ is determined by the density ratio ( as displaced volume is equal to cone weight)[15]

$$\rho_l\,d^3 = \rho_c\,D^3 \Longrightarrow D = d\sqrt[3]{\frac{\rho_l}{\rho_c}} \tag{4.XXVI.e}$$

Substituting equation (4.XXVI.e) into (4.XXVI.d) yield the solution when $\overline{GM} = 0$

$$0 = \frac{\rho_l\,d\,\tan^2\theta}{\rho_s\,192} - \left(\frac{d\sqrt[3]{\frac{\rho_l}{\rho_c}}}{4} - \frac{d}{4}\right) \Longrightarrow \frac{\rho_l\,\tan^2\theta}{\rho_s\,48} = \sqrt[3]{\frac{\rho_l}{\rho_c}} - 1 \tag{4.XXVI.f}$$

Since $\rho_l > \rho_c$ this never happened.

End Solution

To understand these principles consider the following examples.

Example 4.27:

*A solid block of wood of uniform density, $\rho_s = \alpha\,\rho_l$ where ( $0 \le \alpha \le 1$ ) is floating in a liquid. Construct a graph that shows the relationship of the $\overline{GM}$ as a function of ratio height to width. Show that the block's length, $L$, is insignificant for this analysis.*

Solution

Equation (4.158) requires that several quantities should be expressed. The moment of inertia for a block is given in Table 3.1 and is $I_{xx} = \frac{L a^3}{12}$. Where $L$ is the length into the page. The distance $\overline{BG}$ is obtained from Archimedes' theorem and can be expressed as

$$W = \rho_s \overbrace{a\,h\,L}^{V} = \rho_l \overbrace{a\,h_1\,L}^{\text{immersed volume}} \Longrightarrow h_1 = \frac{\rho_s}{\rho_l} h$$

[15]Only the dimension is compared, why?

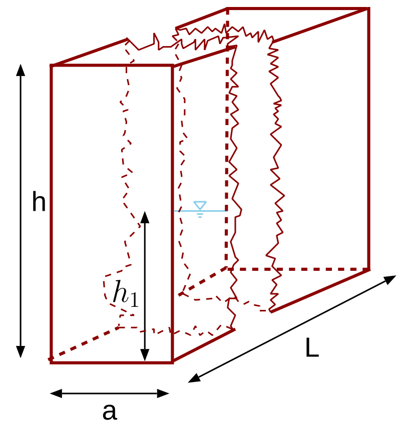



*Fig. -4.39. Cubic body dimensions for stability analysis.*

Thus, the distance $\overline{BG}$ is (see Figure 4.37)

$$\overline{BG} = \frac{h}{2} - \overbrace{\frac{\rho_s}{\rho_l}\,h}^{h_1}\,\frac{1}{2} = \frac{h}{2}\left(1 - \frac{\rho_s}{\rho_l}\right) \tag{4.159}$$

$$GM = \frac{\cancel{g}\,\rho_l\,\dfrac{\overbrace{\cancel{L}\,a^3}^{I_{xx}}}{12}}{\cancel{g}\,\rho_s\,\underbrace{a\,h\,\cancel{L}}_{V}} - \frac{h}{2}\left(1 - \frac{\rho_s}{\rho_l}\right)$$

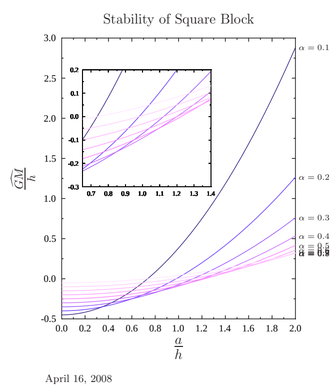



*Fig. -4.40. Stability of cubic body infinity long.*

Simplifying the above equation provides

$$\frac{\overline{GM}}{h} = \frac{1}{12\,\alpha}\left(\frac{a}{h}\right)^2 - \frac{1}{2}\left(1 - \alpha\right) \tag{4.160}$$

where $\alpha$ is the density ratio. Notice that $\overline{GM}/h$ isn't a function of the depth, $L$. This equation leads to the condition where the maximum height above which the body is not stable anymore as

$$\frac{a}{h} \geq \sqrt{6\,(1-\alpha)\alpha} \tag{4.161}$$

End Solution

One of the interesting point for the above analysis is that there is a point above where the ratio of the height to the body width is not stable anymore. In cylindrical shape equivalent to equation (4.161) can be expressed. For cylinder (circle) the moment of inertia is $I_{xx} = \pi\, b^4/64$. The distance $\overline{BG}$ is the same as for the square shape (cubic) (see above (4.159)). Thus, the equation is

$$\frac{\overline{GM}}{h} = \frac{g}{64\,\alpha}\left(\frac{b}{h}\right)^2 - \frac{1}{2}\,(1-\alpha)$$

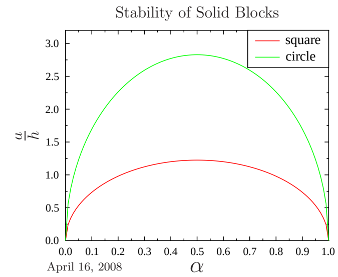



*Fig. -4.41. The maximum height reverse as a function of density ratio.*

And the condition for maximum height for stability is

$$\frac{b}{h} \geq \sqrt{32\,(1-\alpha)\,\alpha}$$

This kind of analysis can be carried for different shapes and the results are shown for these two shapes in Figure 4.41. It can be noticed that the square body is more stable than the circular body shape.

### Principle Main Axises

Any body has infinite number of different axises around which moment of inertia can be calculated. For each of these axises, there is a different moment of inertia. With the exception of the circular shape, every geometrical shape has an axis in which the moment of inertia is without the product of inertia. This axis is where the main rotation of the body will occur. Some analysis of floating bodies are done by breaking the rotation of arbitrary axis to rotate around the two main axises. For stability analysis, it is enough to find if the body is stable around the smallest moment of inertia. For example, a square shape body has larger moment of inertia around diagonal. The difference between the previous calculation and the moment of inertia around the diagonal is

$$\Delta I_{xx} = \overbrace{\frac{\sqrt{2}\,a\left(\frac{\sqrt{3}\,a}{2}\right)^3}{6}}^{I\ diagonal\ axis} - \overbrace{\frac{a^4}{12}}^{\text{``}normal\text{''}axis} \sim 0.07\,a^4$$

Which show that if the body is stable at main axises, it must be stable at the "diagonal" axis. Thus, this problem is reduced to find the stability for principle axis.

### Unstable Bodies

What happen when one increases the height ratio above the maximum height ratio? The body will flip into the side and turn to the next stable point (angle). This is not a hypothetical question, but rather practical. This happens when a ship is overloaded with containers above the maximum height. In commercial ships, the fuel is stored at the bottom of the ship and thus the mass center (point $G$) is changing during the voyage. So, the ship that was stable (positive $\overline{GM}$) leaving the initial port might became unstable (negative $\overline{GM}$) before reaching the destination port.

Example 4.28:

*One way to make a ship to be a hydrodynamic is by making the body as narrow as possible. Suppose that two opposite sides triangle (prism) is attached to each other to create a long "ship" see Figure 4.42. Supposed that $\mathbf{a}/\mathbf{h} \longrightarrow \tilde{0}$ the body will be unstable. On the other side if the $\mathbf{a}/\mathbf{h} \longrightarrow \tilde{\infty}$ the body is very stable. What is the minimum ratio of $\mathbf{a}/\mathbf{h}$ that keep the body stable at half of the volume in liquid (water). Assume that density ratio is $\rho_l/\rho_s = \bar{\rho}$.*

Solution

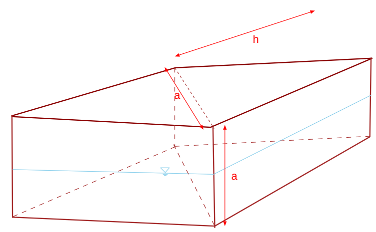



*Fig. -4.42. Stability of two triangles put tougher.*

The answer to the question is that the limiting case where $\overline{GM} = 0$. To find this ratio equation terms in (4.158) have to be found. The Volume of the body is

$$V = 2\left(\frac{a^2\,h}{2}\right) = a^2\,h$$

The moment of inertia is triangle (see explanation in example (3.7) is

$$I_{xx} = \frac{a\,h^3}{2}$$

And the volume is

$$V_{body} = a^2\,\sqrt{h^2 - \frac{a^2}{4}} = a^2\,h\,\sqrt{1 - \frac{1}{4}\,\frac{a^2}{h^2}}$$

The point $\mathbf{B}$ is a function of the density ratio of the solid and liquid. Denote the liquid density as $\rho_l$ and solid density as $\rho_s$. The point $\mathbf{B}$ can be expressed as

$$B = \frac{a\,\rho_s}{2\,\rho_l}$$

And thus the distance $\overline{BG}$ is

$$\overline{BG} = \frac{a}{2}\left(1 - \frac{\rho_s}{\rho_l}\right)$$

The limiting condition requires that $\overline{GM} = 0$ so that

$$\frac{\rho_l\,I_{xx}}{\rho_s V_{body}} = \overline{BG}$$

Or explicitly

$$\frac{\rho_l \dfrac{a\,h^3}{2}}{\rho_s\,a^2\,h\,\sqrt{1-\dfrac{1}{4}\,\dfrac{a^2}{h^2}}} = \frac{a}{2}\left(1-\frac{\rho_s}{\rho_l}\right)$$

After rearrangement and using the definitions of $\xi = h/a\ \bar{\rho}\rho_l/\rho_s$ results in

$$\frac{\bar{\rho}\,\xi^2}{\sqrt{1-\dfrac{\xi^2}{4}}} = \left(1-\frac{1}{\bar{\rho}}\right)$$

The solution of the above solution is obtained by squaring both sides and defining a new variable such as $x = \xi^2$. After the above manipulation and selecting the positive value and to keep stability as

$$x < \frac{\sqrt{\frac{\sqrt{64\,\bar{\rho}^4-64\,\bar{\rho}^3+\bar{\rho}^2-2\,\bar{\rho}+1}}{\bar{\rho}}+\frac{1}{\bar{\rho}}-1}}{2\sqrt{2}\,\bar{\rho}}$$

End Solution

#### 4.6.1.1 Stability of Body with Shifting Mass Centroid

Ships and other floating bodies carry liquid or have a load which changes the mass location during tilting of the floating body. For example, a ship that carries wheat grains where the cargo is not properly secured to the ship. The movement of the load (grains, furniture, and/or liquid) does not occur in the same speed as the body itself or the displaced outside liquid. Sometimes, the slow reaction of the load, for stability analysis, is enough to be ignored. Exact analysis requires taking into account these shifting mass speeds. However, here, the extreme case where the load reacts in the same speed as the tilting of the ship/floating body is examined. For practical purposes, it is used as a limit for the stability analysis. There are situations where the real case approaches to this extreme. These situations involve liquid with a low viscosity (like water, alcohol) and ship with low natural frequency (later on the frequency of the ships). Moreover, in this analysis, the dynamics are ignored and only the statics is examined (see Figure 4.43).

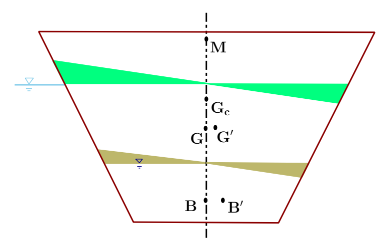



*Fig. -4.43. The effects of liquid movement on the $\overline{GM}$.*

A body is loaded with liquid "B" and is floating in a liquid "A" as shown in Figure 4.43. When the body is given a tilting position the body displaces the liquid on the outside. At the same time, the liquid inside is changing its mass centroid. The moment created by the inside displaced liquid is

$$M_{in} = g\,\rho_{l\,B}\beta I_{xx\,B} \tag{4.162}$$

Note that $I_{xx\,B}$ isn't the same as the moment of inertia of the outside liquid interface.

The change in the mass centroid of the liquid "A" then is

$$\overline{G_1G_1'} = \frac{\cancel{g}\,\cancel{\rho_{l\,B}}\beta I_{xx\,B}}{\underbrace{\cancel{g}\,V_B\,\cancel{\rho_{l\,B}}}_{\text{Inside liquid weight}}} = \frac{I_{xx\,B}}{V_B} \tag{4.163}$$

Equation (4.163) shows that $\overline{GG'}$ is only a function of the geometry. This quantity, $\overline{G_1G_1'}$, is similar for all liquid tanks on the floating body.

The total change of the vessel is then calculated similarly to center area calculations.

$$\cancel{g}\,m_{total}\,\overline{GG'} = \cancelto{0}{g\,m_{body}} + \cancel{g}\,m_f\overline{G_1G_1'} \tag{4.164}$$

For more than one tank, it can be written as

$$\overline{GG'} = \frac{g}{W_{total}}\sum_{i=1}^{n}\overline{G_iG_i}\rho_{l_i}V_i = \frac{g}{W_{total}}\sum_{i=1}^{n}\frac{I_{xx\,b\,i}}{V_{b\,i}} \tag{4.165}$$

A new point can be defined as $G_c$. This point is the intersection of the center line with the vertical line from $G'$.

$$\overline{G\,G_c} = \frac{\overline{GG'}}{\sin\beta} \tag{4.166}$$

The distance that was used before $\overline{GM}$ is replaced by the criterion for stability by $\overline{G_c\,M}$ and is expressed as

$$\overline{G_c\,M} = \frac{g\,\rho_A\,I_{xx\,A}}{\rho_s V_{body}} - \overline{BG} - \frac{1}{m_{total}}\,\frac{I_{xx\,b}}{V_b} \tag{4.167}$$

If there are more than one tank partially filled with liquid, the general formula is

$$\overline{G_c\,M} = \frac{g\,\rho_A\,I_{xx\,A}}{\rho_s V_{body}} - \overline{BG} - \frac{1}{m_{total}}\sum_{i=1}^{n}\frac{I_{xx\,b\,i}}{V_{b\,i}} \tag{4.168}$$

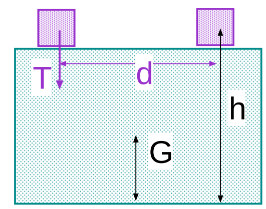


*Fig. -4.44. Measurement of GM of floating body.*

One way to reduce the effect of the moving mass center due to liquid is done by substituting a single tank with several tanks. The moment of inertial of the combine two tanks is smaller than the moment of inertial of a single tank. Increasing the number of tanks reduces the moment of inertia. The engineer could design the tanks in such a way that the moment of inertia is operationally changed. This control of the stability, $\overline{GM}$, can be achieved by having some tanks spanning across the entire body with tanks spanning on parts of the body. Movement of the liquid (mostly the fuel and water) provides way to control the stability, $GM$, of the ship.

#### 4.6.1.2 Metacentric Height, $\overline{GM}$, Measurement

The metacentric height can be measured by finding the change in the angle when a weight is moved on the floating body.

Moving the weight, $T$ a distance, $d$ then the moment created is

$$M_{weight} = T\,d \tag{4.169}$$

This moment is balanced by

$$M_{righting} = W_{total}\overline{GM}_{new}\,\theta \tag{4.170}$$

Where, $W_{total}$, is the total weight of the floating body including measuring weight. The angle, $\theta$, is measured as the difference in the orientation of the floating body. The metacentric height is

$$\overline{GM}_{new} = \frac{T\,d}{W_{total}\,\theta} \tag{4.171}$$

If the change in the $\overline{GM}$ can be neglected, equation (4.171) provides the solution. The calculation of $\overline{GM}$ can be improved by taking into account the effect of the measuring weight. The change in height of $G$ is

$$\cancel{g}\,m_{total}\,G_{new} = \cancel{g}\,m_{ship}\,G_{actual} + \cancel{g}\,T\,h \tag{4.172}$$

Combining equation (4.172) with equation (4.171) results in

$$\overline{GM}_actual = \overline{GM}_{new}\,\frac{m_{total}}{m_{ship}} - h\,\frac{T}{m_{ship}} \tag{4.173}$$

The weight of the ship is obtained from looking at the ship depth.

#### 4.6.1.3 Stability of Submerged Bodies

The analysis of submerged bodied is different from the stability when the body lays between two fluid layers with different density. When the body is submerged in a single fluid layer, then none of the changes of buoyant centroid occurs. Thus, the mass centroid must be below than buoyant centroid in order to have stable condition.

However, all fluids have density varied in some degree. In cases where the density changes significantly, it must be taken into account. For an example of such a case is an object floating in a solar pond where the upper layer is made of water with lower salinity than the bottom layer(change up to 20% of the density). When the floating object is immersed into two layers, the stability analysis must take into account the changes of the displaced liquids of the two liquid layers. The calculations for such cases are a bit more complicated but based on the similar principles. Generally, this density change helps to increase the stability of the floating bodies. This analysis is out of the scope of this book (for now).

#### 4.6.1.4 Stability of None Systematical or "Strange" Bodies

While most floating bodies are symmetrical or semi–symmetrical, there are situations where the body has a "strange" and/or un-symmetrical body. Consider the first strange body that has an abrupt step change as shown in Figure 4.45. The body weight doesn't change during the rotation that the green area on the left and the green area on right are the same (see Figure 4.45). There are two situations that can occur. After the tilting, the upper part of the body is above the liquid or part of the body is submerged under the water. The mathematical condition for the border is when $b = 3\,a$. For the case of $b < 3\,a$ the calculation of moment of inertia are similar to the previous case. The moment created by change in the displaced liquid (area) act in the same fashion as the before. The center of the moment is needed to be found. This point is the intersection of the liquid line with the brown middle line. The moment of inertia should be calculated around this axis.

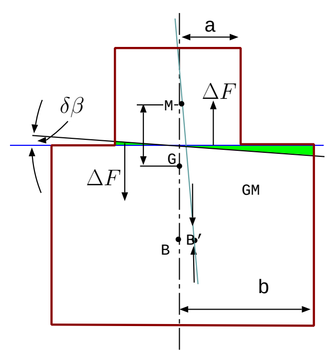



*Fig. -4.45. Calculations of $\overline{GM}$ for abrupt shape body.*

For the case where $b < 3\,a$ $x$ some part is under the liquid. The amount of area under the liquid section depends on the tilting angle. These calculations are done as if none of the body under the liquid. This point is intersection point liquid with lower body and it is needed to be calculated. The moment of inertia is calculated around this point (note the body is "ended" at end of the upper body). However, the moment to return the body is larger than actually was calculated and the bodies tend to be more stable (also for other reasons).

#### 4.6.1.5 Neutral frequency of Floating Bodies

This case is similar to pendulum (or mass attached to spring). The governing equation for the pendulum is

$$\ell\ddot{\beta} - g\,\beta = 0 \tag{4.174}$$

Where here $\ell$ is length of the rode (or the line/wire) connecting the mass with the rotation point. Thus, the frequency of pendulum is $\frac{1}{2\,\pi}\sqrt{\frac{g}{\ell}}$ which measured in $Hz$. The period of the cycle is $2\,\pi\,\sqrt{\ell/g}$. Similar situation exists in the case of floating bodies. The basic differential equation is used to balance and is

$$\overbrace{I\ddot{\beta}}^{rotation} \;-\; \overbrace{V\,\rho_s\,\overline{GM}\,\beta}^{rotating\ moment} \;= 0 \tag{4.175}$$

In the same fashion the frequency of the floating body is

$$\frac{1}{2\,\pi}\sqrt{\frac{V\,\rho_s\,\overline{GM}}{I_{body}}} \tag{4.176}$$

and the period time is

$$2\,\pi\sqrt{\frac{I_{body}}{V\,\rho_s\,\overline{GM}}} \tag{4.177}$$

In general, the larger $\overline{GM}$ the more stable the floating body is. Increase in $\overline{GM}$ increases the frequency of the floating body. If the floating body is used to transport humans and/or other creatures or sensitive cargo it requires to reduce the $\overline{GM}$ so that the traveling will be smoother.

### 4.6.2 Surface Tension

The surface tension is one of the mathematically complex topic and related to many phenomena like boiling, coating, etc. In this section, only simplified topics like constant value will be discussed.

Example 4.29:
*In interaction of the molecules shown in Figure ? describe the existence of surface tension. Explain why this description is erroneous?*

SOLUTION

The upper layer of the molecules have unbalanced force towards the liquid phase. Newton's law states when there is unbalanced force, the body should be accelerate. However, in this case, the liquid is not in motion. Thus, the common explanation is wrong.

End Solution

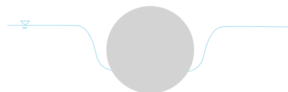


*Fig. -4.46. A heavy needle is floating on a liquid.*

Example 4.30:

*Needle is made of steel and is heavier than water and many other liquids. However, the surface tension between the needle and the liquid hold the needle above the liquid. After certain diameter, the needle cannot be held by the liquid. Calculate the maximum diameter needle that can be inserted into liquid without drowning.*

SOLUTION

Under Construction

End Solution

## 4.7 Rayleigh–Taylor Instability

RayleighTaylor instability (or RT instability) is named after Lord Rayleigh and G. I. Taylor. There are situations where a heavy liquid layer is placed over a lighter fluid layer. This situation has engineering implications in several industries. For example in die casting, liquid metal is injected in a cavity filled with air. In poor designs or other situations, some air is not evacuated and stay in small cavity on the edges of the shape to be casted. Thus, it can create a situation where the liquid metal is above the air but cannot penetrate into the cavity because of instability.

This instability deals with a dense, heavy fluid that is being placed above a lighter fluid in a gravity field perpendicular to interface. Example for such systems are dense water over oil (liquid–liquid), or water over air(gas–liquid). The original Rayleigh's paper deals with the dynamics and density variations. For example, density variations according to the bulk modulus (see section 4.3.3.2) are always stable but unstable of the density is in the reversed order.

Supposed that a liquid density is arbitrary function of the height. This distortion can be as a result of heavy fluid above the lighter liquid. This analysis asks the question of what happen when a small amount of liquid from the above layer enter into the lower layer? Whether this liquid continue and will grow or will it return to its original conditions? The surface tension is the opposite mechanism that will returns the liquid to its original place. This analysis is referred to the case of infinite or very large surface. The simplified case is the two different uniform densities. For example a heavy fluid density, $\rho_L$, above lower fluid with lower density, $\rho_G$.

For perfectly straight interface, the heavy fluid will stay above the lighter fluid. If the surface will be disturbed, some of heavy liquid moves down. This disturbance can grow or returned to its original situation. This condition is determined by competing forces, the surface density, and the buoyancy forces. The fluid above the depression is in equilibrium with the sounding pressure since the material is extending to infinity. Thus, the force that acting to get the above fluid down is the buoyancy force of the fluid in the depression.

The depression is returned to its original position if the surface forces are large enough. In that case, this situation is considered to be stable. On the other hand, if the surface forces (surface tension) are not sufficient, the situation is unstable and the heavy liquid enters into the liquid fluid zone and vice versa. As usual there is the neutral stable when the forces are equal. Any continues function can be expanded in series of cosines. Thus, example of a cosine function will be examined. The conditions that required from this function will be required from all the other functions. The disturbance is of the following

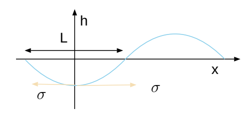


*Fig. -4.47. Description of depression to explain the Rayleigh–Taylor instability.*

$$h = -h_{max} \cos \frac{2\,\pi\,x}{L} \tag{4.178}$$

where $h_{max}$ is the maximum depression and $L$ is the characteristic length of the depression. The depression has different radius as a function of distance from the center of the depression, $x$. The weakest point is at $x = 0$ because symmetrical reasons the surface tension does not act against the gravity as shown in Figure (4.47). Thus, if the center point of the depression can "hold" the intrusive fluid then the whole system is stable.

The radius of any equation is expressed by equation (1.57). The first derivative of $\cos$ around zero is $\sin$ which is approaching zero or equal to zero. Thus, equation (1.57) can be approximated as

$$\frac{1}{R} = \frac{d^2h}{dx^2} \tag{4.179}$$

For equation (4.178) the radius is

$$\frac{1}{R} = -\frac{4\,\pi^2\,h_{max}}{L^2} \tag{4.180}$$

According to equation (1.46) the pressure difference or the pressure jump is due to the surface tension at this point must be

$$P_H - P_L = \frac{4\,h_{max}\,\sigma\,\pi^2}{L^2} \tag{4.181}$$

The pressure difference due to the gravity at the edge of the disturbance is then

$$P_H - P_L = g\,\left(\rho_H - \rho_L\right) h_{max} \tag{4.182}$$

Comparing equations (4.181) and (4.182) show that if the relationship is

$$\frac{4\,\sigma\,\pi^2}{L^2} > g\,(\rho_H - \rho_L) \tag{4.183}$$

It should be noted that $h_{max}$ is irrelevant for this analysis as it is canceled. The point where the situation is neutral stable

$$L_c = \sqrt{\frac{4\,\pi^2\sigma}{g\,(\rho_H - \rho_L)}} \tag{4.184}$$

An alternative approach to analyze this instability is suggested here. Consider the situation described in Figure 4.48. If all the heavy liquid "attempts" to move straight down, the lighter liquid will "prevent" it. The lighter liquid needs to move up at the same time but in a different place. The heavier liquid needs to move in one side and the lighter liquid in another location. In this process the heavier liquid "enter" the lighter liquid in one point and creates a depression as shown in Figure 4.48.

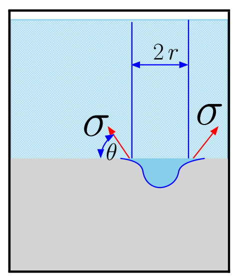



*Fig. -4.48. Description of depression to explain the instability.*

To analyze it, considered two control volumes bounded by the blue lines in Figure 4.48. The first control volume is made of a cylinder with a radius $r$ and the second is the depression below it. The "extra" lines of the depression should be ignored, they are not part of the control volume. The horizontal forces around the control volume are canceling each other. At the top, the force is atmospheric pressure times the area. At the cylinder bottom, the force is $\rho\,g\,h \times A$. This acts against the gravity force which make the cylinder to be in equilibrium with its surroundings if the pressure at bottom is indeed $\rho\,g\,h$.

For the depression, the force at the top is the same force at the bottom of the cylinder. At the bottom, the force is the integral around the depression. It can be approximated as a flat cylinder that has depth of $r\,\pi/4$ (read the explanation in the example 4.21) This value is exact if the shape is a perfect half sphere. In reality, the error is not significant. Additionally when the depression occurs, the liquid level is reduced a bit and the lighter liquid is filling the missing portion. Thus, the force at the bottom is

$$F_{bottom} \sim \pi\,r^2 \left[\left(\frac{\pi\,r}{4} + h\right)\,(\rho_L - \rho_G)\;g + P_{atmos}\right] \tag{4.185}$$

The net force is then

$$F_{bottom} \sim \pi\,r^2 \left(\frac{\pi\,r}{4}\right)\,(\rho_L - \rho_G)\;g \tag{4.186}$$

The force that hold this column is the surface tension. As shown in Figure 4.48, the total force is then

$$F_\sigma = 2\,\pi\,r\,\sigma\,\cos\theta \tag{4.187}$$

The forces balance on the depression is then

$$2\,\pi\,r\,\sigma\cos\theta \sim \pi\,r^2\left(\frac{\pi\,r}{4}\right)\left(\rho_L-\rho_G\right)\,g \tag{4.188}$$

The radius is obtained by

$$r \sim \sqrt{\frac{2\,\pi\,\sigma\cos\theta}{\left(\rho_L-\rho_G\right)\,g}} \tag{4.189}$$

The maximum surface tension is when the angle, $\theta=\pi/2$. At that case, the radius is

$$r \sim \sqrt{\frac{2\,\pi\,\sigma}{\left(\rho_L-\rho_G\right)\,g}} \tag{4.190}$$

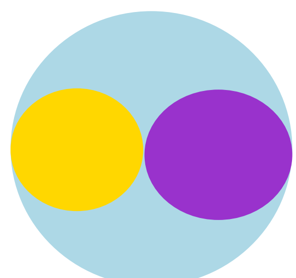


*Fig. -4.49. The cross section of the interface. The purple color represents the maximum heavy liquid raising area. The yellow color represents the maximum lighter liquid that are "going down."*

The maximum possible radius of the depression depends on the geometry of the container. For the cylindrical geometry, the maximum depression radius is about half for the container radius (see Figure 4.49). This radius is limited because the lighter liquid has to enter at the same time into the heavier liquid zone. Since the "exchange" volumes of these two process are the same, the specific radius is limited. Thus, it can be written that the minimum radius is

$$r_{mintube} = 2\sqrt{\frac{2\,\pi\,\sigma}{g\left(\rho_L-\rho_G\right)}} \tag{4.191}$$

The actual radius will be much larger. The heavier liquid can stay on top of the lighter liquid without being turned upside down when the radius is smaller than the equation 4.191. This analysis introduces a new dimensional number that will be discussed in a greater length in the Dimensionless chapter. In equation (4.191) the angle was assumed to be 90 degrees. However, this angle is never can be obtained. The actual value of this angle is about $\pi/4$ to $\pi/3$ and in only extreme cases the angle exceed this value (considering dynamics). In Figure 4.49, it was shown that the depression and the raised area are the same. The actual area of the depression is only a fraction of the interfacial cross section and is a function. For example,the depression is larger for square area. These two scenarios should be inserting into equation 4.168 by introducing experimental coefficient.

Example 4.31:

*Estimate the minimum radius to insert liquid aluminum into represent tube at temperature of 600*$[K]$*. Assume that the surface tension is* $400[mN/m]$*. The density of the aluminum is* $2400 kg/m^3$*.*

Solution

The depression radius is assume to be significantly smaller and thus equation (4.190) can be used. The density of air is negligible as can be seen from the temperature compare to the aluminum density.

$$r \sim \sqrt{\frac{8\,\pi\, \overbrace{0.4}^{\sigma}}{2400 \times 9.81}}$$

The minimum radius is $r \sim 0.02[m]$ which demonstrates the assumption of $h >> r$ was appropriate.

End Solution

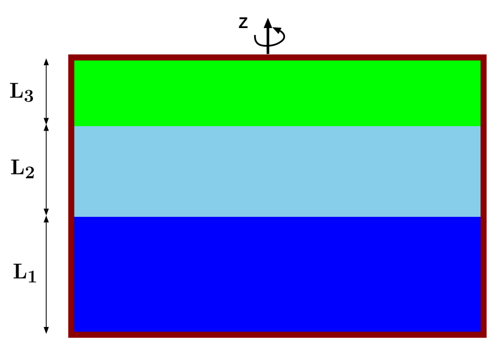



*Fig. -4.50. Three liquids layers under rotation with various critical situations.*

**Open Question by April 15, 2010**

The best solution of the following question will win 18 U.S. dollars and your name will be associated with the solution in this book.

Example 4.32:

*A canister shown in Figure 4.50 has three layers of different fluids with different densities. Assume that the fluids do not mix. The canister is rotate with circular velocity, $\omega$. Describe the interface of the fluids consider all the limiting cases. Is there any difference if the fluids are compressible? Where is the maximum pressure points? For the case that the fluids are compressible, the canister top center is connected to another tank with equal pressure to the canister before the rotation (the connection point). What happen after the canister start to be rotated? Calculated the volume that will enter or leave, for known geometries of the fluids. Use the ideal gas model. You can assume that the process is isothermal. Is there any difference if the process is isentropic? If so, what is the difference?*

# Part I

# Integral Analysis

# CHAPTER 5

# The Control Volume and Mass Conservation

## 5.1 Introduction

This chapter presents a discussion on the control volume and will be focused on the conservation of the mass. When the fluid system moves or changes, one wants to find or predict the velocities in the system. The main target of such analysis is to find the value of certain variables. This kind of analysis is reasonable and it referred to in the literature as the Lagrangian Analysis. This name is in honored J. L. Langrange (1736–1813) who formulated the equations of motion for the moving fluid particles.

Even though this system looks reasonable, the Lagrangian system turned out to be difficult to solve and to analyze. This method applied and used in very few cases. The main difficulty lies in the fact that every particle has to be traced to its original state. Leonard Euler (1707–1783) suggested an alternative approach. In Euler's approach the focus is on a defined point or a defined volume. This methods is referred as Eulerian method.

The Eulerian method focuses on a defined area or location to find the needed information. The use of the Eulerian methods leads to a set differentiation equations that is referred to as Navier–Stokes equations which are commonly used. These differential equations will be used in the later part of this book. Ad-

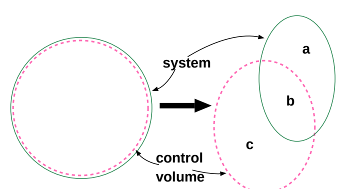



*Fig. -5.1. Control volume and system before and after motion.*

ditionally, the Eulerian system leads to integral equations which are the focus of this part of the book. The Eulerian method plays well with the physical intuition of most people. This methods has its limitations and for some cases the Lagrangian is preferred (and sometimes the only possibility). Therefore a limited discussion on the Lagrangian system will be presented (later version).

Lagrangian equations are associated with the system while the Eulerian equation are associated with the control volume. The difference between the system and the control volume is shown in Figure 5.1. The green lines in Figure 5.1 represent the system. The red dotted lines are the control volume. At certain time the system and the control volume are identical location. After a certain time, some of the mass in the system exited the control volume which are marked "**a**" in Figure 5.1. The material that remained in the control volume is marked as "**b**". At the same time, the control gains some material which is marked as "**c**".

## 5.2 Control Volume

The Eulerian method requires to define a control volume (some time more than one). The control volume is a defined volume that was discussed earlier. The control volume is differentiated into two categories of control volumes, non–deformable and deformable.

**Non–deformable control volume** is a control volume which is fixed in space relatively to an one coordinate system. This coordinate system may be in a relative motion to another (almost absolute) coordinate system.

**Deformable control volume** is a volume having part of all of its boundaries in motion during the process at hand.

In the case where no mass crosses the boundaries, the control volume is a system. Every control volume is the focus of the certain interest and will be dealt with the basic equations, mass, momentum, energy, entropy etc.

Two examples of control volume are presented to illustrate difference between a deformable control volume and non–deformable control volume. Flow in conduits can be analyzed by looking in a control volume between two locations. The coordinate system could be fixed to the conduit. The control volume chosen is non-deformable control volume. The control volume should be chosen so that the analysis should be simple and dealt with as less as possible issues which are not in question. When a piston pushing gases a good choice of control volume is a deformable control volume that is a head the piston inside the cylinder as shown in Figure 5.2.

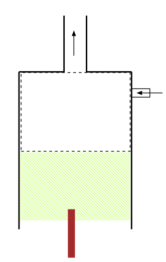


*Fig. -5.2. Control volume of a moving piston with in and out flow.*

## 5.3 Continuity Equation

In this chapter and the next three chapters, the conservation equations will be applied to the control volume. In this chapter, the mass conservation will be discussed. The system mass change is

$$\frac{D\,m_{sys}}{Dt} = \frac{D}{Dt}\int_{V_{sys}} \rho dV = 0 \tag{5.1}$$

The system mass after some time, according Figure 5.1, is made of

$$m_{sys} = m_{c.v.} + m_a - m_c \tag{5.2}$$

The change of the system mass is by definition is zero. The change with time (time derivative of equation (5.2)) results in

$$0 = \frac{D\,m_{sys}}{Dt} = \frac{d\,m_{c.v.}}{dt} + \frac{d\,m_a}{dt} - \frac{d\,m_c}{dt} \tag{5.3}$$

The first term in equation (5.3) is the derivative of the mass in the control volume and at any given time is

$$\frac{d\,m_{c.v.}(t)}{dt} = \frac{d}{dt}\int_{V_{c.v.}} \rho\, dV \tag{5.4}$$

and is a function of the time.

The interface of the control volume can move. The actual velocity of the fluid leaving the control volume is the relative velocity (see Figure 5.3). The relative velocity is

$$\overrightarrow{U_r} = \overrightarrow{U_f} - \overrightarrow{U_b} \tag{5.5}$$

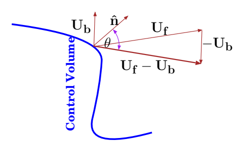



*Fig. -5.3. Schematics of velocities at the interface.*

Where $U_f$ is the liquid velocity and $U_b$ is the boundary velocity (see Figure 5.3). The velocity component that is perpendicular to the surface is

$$U_{rn} = -\hat{n}\cdot\overrightarrow{U_r} = U_r\,\cos\theta \tag{5.6}$$

Where $\hat{n}$ is an unit vector perpendicular to the surface. The convention of direction is taken positive if flow out the control volume and negative if the flow is into the control volume. The mass flow out of the control volume is the system mass that is not included in the control volume. Thus, the flow out is

$$\frac{d\,m_a}{dt} = \int_{S_{cv}} \rho_s\, U_{rn} dA \tag{5.7}$$

It has to be emphasized that the density is taken at the surface thus the subscript $s$. In the same manner, the flow rate in is

$$\frac{d\,m_b}{dt} = \int_{S_{c.v.}} \rho_s\, U_{rn} dA \tag{5.8}$$

It can be noticed that the two equations (5.8) and (5.7) are similar and can be combined, taking the positive or negative value of $U_{rn}$ with integration of the entire system as

$$\frac{d\,m_a}{dt} - \frac{d\,m_b}{dt} = \int_{S_{cv}} \rho_s\, U_{rn}\, dA \tag{5.9}$$

applying negative value to keep the convention. Substituting equation (5.9) into equation (5.3) results in

Continuity

$$\frac{d}{dt}\int_{c.v.} \rho_s dV = -\int_{S_{cv}} \rho\, U_{rn}\, dA \tag{5.10}$$

Equation (5.10) is essentially accounting of the mass. Again notice the negative sign in surface integral. The negative sign is because flow out marked positive which reduces of the mass (negative derivative) in the control volume. The change of mass change inside the control volume is net flow in or out of the control system.

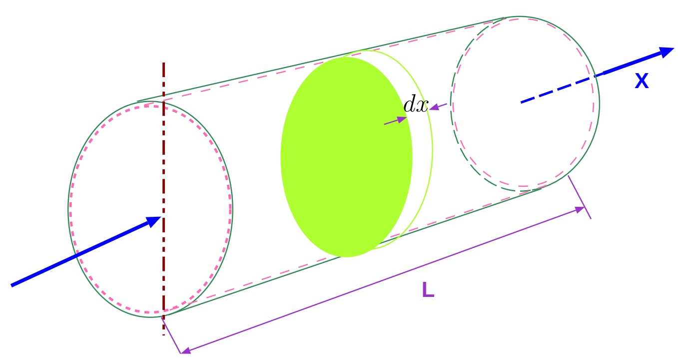



*Fig. -5.4. Schematics of flow in in pipe with varying density as a function time for example 5.1.*

The next example is provided to illustrate this concept.

Example 5.1:

*The density changes in a pipe, due to temperature variation and other reasons, can be approximated as*

$$\frac{\rho(x,t)}{\rho_0} = \left(1 - \frac{x}{L}\right)^2 \cos\frac{t}{t_0}.$$

*The conduit shown in Figure 5.4 length is $L$ and its area is $A$. Express the mass flow in and/or out, and the mass in the conduit as function of time. Write the expression for the mass change in the pipe.*

Solution

Here it is very convenient to choose a non-deformable control volume that is inside the conduit $dV$ is chosen as $\pi\, R^2\, dx$. Using equation (5.10), the flow out (or in) is

$$\frac{d}{dt}\int_{c.v.}\rho dV = \frac{d}{dt}\int_{c.v.} \overbrace{\rho_0\left(1-\frac{x}{L}\right)^2\cos\left(\frac{t}{t_0}\right)}^{\rho(t)} \overbrace{\pi\, R^2\, dx}^{dV}$$

The density is not a function of radius, $r$ and angle, $\theta$ and they can be taken out the integral as

$$\frac{d}{dt}\int_{c.v.}\rho dV = \pi\, R^2 \frac{d}{dt}\int_{c.v.} \rho_0\left(1-\frac{x}{L}\right)^2\cos\left(\frac{t}{t_0}\right) dx$$

which results in

$$\textbf{Flow Out} = \overbrace{\pi\, R^2}^{A} \frac{d}{dt}\int_0^L \rho_0\left(1-\frac{x}{L}\right)^2 \cos\frac{t}{t_0} dx = -\frac{\pi\, R^2\, L\, \rho_0}{3\, t_0}\sin\left(\frac{t}{t_0}\right)$$

The flow out is a function of length, $L$, and time, $t$, and is the change of the mass in the control volume.

End Solution

## 5.3.1 Non Deformable Control Volume

When the control volume is fixed with time, the derivative in equation (5.10) can enter the integral since the boundaries are fixed in time and hence,

Continuity with Fixed b.c.

$$\int_{V_{c.v.}} \frac{d\,\rho}{dt} dV = -\int_{S_{c.v.}} \rho\, U_{rn}\, dA \tag{5.11}$$

Equation (5.11) is simpler than equation (5.10).

## 5.3.2 Constant Density Fluids

Further simplifications of equations (5.10) can be obtained by assuming constant density and the equation (5.10) become conservation of the volume.

#### 5.3.2.1 Non Deformable Control Volume

For this case the volume is constant therefore the mass is constant, and hence the mass change of the control volume is zero. Hence, the net flow (in and out) is zero. This condition can be written mathematically as

$$\frac{\overbrace{d\int}^{=0}}{dt} \longrightarrow \int_{S_{c.v.}} V_{rn} dA = 0 \tag{5.12}$$

or in a more explicit form as

Steady State Continuity

$$\int_{S_{in}} V_{rn}\, dA = \int_{S_{out}} V_{rn}\, dA = 0 \tag{5.13}$$

Notice that the density does not play a role in this equation since it is canceled out. Physically, the meaning is that volume flow rate in and the volume flow rate out have to equal.

#### 5.3.2.2 Deformable Control Volume

The left hand side of question (5.10) can be examined further to develop a simpler equation by using the extend Leibniz integral rule for a constant density and result in

$$\frac{d}{dt}\int_{c.v.} \rho\, dV = \overbrace{\int_{c.v.} \frac{\overbrace{d\,\rho}^{=0}}{dt}}^{\text{thus, }=0} dV + \rho \int_{S_{c.v.}} \hat{n}\cdot U_b\, dA = \rho \int_{S_{c.v.}} U_{bn}\, dA \tag{5.14}$$

where $U_b$ is the boundary velocity and $U_{bn}$ is the normal component of the boundary velocity.

Steady State Continuity Deformable

$$\int_{S_{c.v.}} U_{bn}\, dA = \int_{S_{c.v.}} U_{rn}\, dA \tag{5.15}$$

The meaning of the equation (5.15) is the net growth (or decrease) of the Control volume is by net volume flow into it. Example 5.2 illustrates this point.

Example 5.2:

*Liquid fills a bucket as shown in Figure 5.5. The average velocity of the liquid at the exit of the filling pipe is $U_p$ and cross section of the pipe is $A_p$. The liquid fills a bucket with cross section area of $\mathbf{A}$ and instantaneous height is $h$. Find the height as a function of the other parameters. Assume that the density is constant and at the boundary interface $A_j = 0.7\, A_p$. And where $A_j$ is the area of jet when touching the*

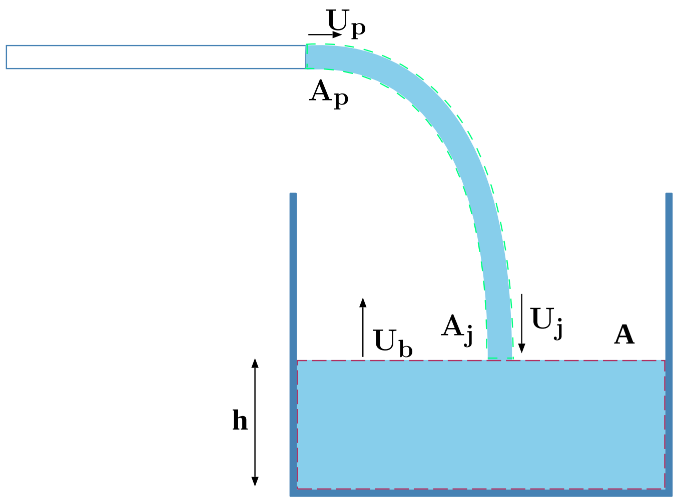



*Fig. -5.5. Filling of the bucket and choices of the deformable control volumes for example 5.2.*

*liquid boundary in bucket. The last assumption is result of the energy equation (with some influence of momentum equation). The relationship is function of the distance of the pipe from the boundary of the liquid. However, this effect can be neglected for this range which this problem. In reality, the ratio is determined by height of the pipe from the liquid surface in the bucket. Calculate the bucket liquid interface velocity.*

Solution

This problem requires two deformable control volumes. The first control is around the jet and second is around the liquid in the bucket. In this analysis, several assumptions must be made. First, no liquid leaves the jet and enters the air. Second, the liquid in the bucket has a straight surface. This assumption is a strong assumption for certain conditions but it will be not discussed here since it is advance topic. Third, there are no evaporation or condensation processes. Fourth, the air effects are negligible. The control volume around the jet is deformable because the length of the jet shrinks with the time. The mass conservation of the liquid in the bucket is

$$\overbrace{\int_{c.v.} U_{bn}\, dA}^{\text{boundary change}} = \overbrace{\int_{c.v.} U_{rn}\, dA}^{\text{flow in}}$$

where $U_{bn}$ is the perpendicular component of velocity of the boundary. Substituting the known values for $U_{rn}$ results in

$$\int_{c.v.} U_b\, dA = \int_{c.v.} \overbrace{(U_j + U_b)}^{U_{rn}}\, dA$$

The integration can be carried when the area of jet is assumed to be known as

$$U_b\, A = A_j\, (U_j + U_b) \tag{5.II.a}$$

To find the jet velocity, $U_j$, the second control volume around the jet is used as the following

$$\overbrace{U_p\, A_p}^{\text{flow in}} - \overbrace{A_j\, (U_b + U_j)}^{\text{flow out}} = \overbrace{-A_j\, U_b}^{\text{boundary change}} \tag{5.II.b}$$

The above two equations (5.II.a) and (5.II.b) are enough to solve for the two unknowns. Substituting the first equation, (5.II.a) into (5.II.b) and using the ratio of $A_j = 0.7\, A_p$ results

$$U_p\, A_p - U_b\, A = -0.7\, A_p\, U_b \tag{5.II.c}$$

The solution of equation (5.II.c) is

$$U_b = \frac{A_p}{A - 0.7\, A_p}$$

It is interesting that many individuals intuitively will suggest that the solution is $U_b A_p/A$. When examining solution there are two limits. The first limit is when $A_p = A/0.7$ which is

$$U_b = \frac{A_p}{0} = \infty$$

The physical meaning is that surface is filled instantly. The other limit is that and $A_p/A \longrightarrow 0$ then

$$U_b = \frac{A_p}{A}$$

which is the result for the "intuitive" solution. It also interesting to point out that if the filling was from other surface (not the top surface), e.g. the side, the velocity will be $U_b = U_p$ in the limiting case and not infinity. The reason for this difference is that the liquid already fill the bucket and has not to move into bucket.

End Solution

Example 5.3:

*Balloon is attached to a rigid supply in which is supplied by a constant the mass rate, $m_i$. Calculate the velocity of the balloon boundaries assuming constant density.*

Solution

The applicable equation is

$$\int_{c.v.} U_{bn}\, dA = \int_{c.v.} U_{rn}\, dA$$

The entrance is fixed, thus the relative velocity, $U_{rn}$ is

$$U_{rn} = \begin{cases} -U_p & \text{@ the valve} \\ 0 & \text{every else} \end{cases}$$

Assume equal distribution of the velocity in balloon surface and that the center of the balloon is moving, thus the velocity has the following form

$$U_b = U_x\,\hat{x} + U_{br}\,\hat{r}$$

Where $\hat{x}$ is unit coordinate in $x$ direction and $U_x$ is the velocity of the center and where $\hat{r}$ is unit coordinate in radius from the center of the balloon and $U_{br}$ is the velocity in that direction. The right side of equation (5.15) is the net change due to the boundary is

$$\int_{S_{c.v.}} \left(U_x\,\hat{x} + U_{br}\,\hat{r}\right)\cdot\hat{n}\,dA = \overbrace{\int_{S_{c.v.}} \left(U_x\,\hat{x}\right)\cdot\hat{n}\,dA}^{\text{center movement}} + \overbrace{\int_{S_{c.v.}} \left(U_{br}\,\hat{r}\right)\cdot\hat{n}\,dA}^{\text{net boundary change}}$$

The first integral is zero because it is like movement of solid body and also yield this value mathematically (excises for mathematical oriented student). The second integral (notice $\hat{n} = \hat{r}$) yields

$$\int_{S_{c.v.}} \left(U_{br}\,\hat{r}\right)\cdot\hat{n}\,dA = 4\,\pi\,r^2\,U_{br}$$

Substituting into the general equation yields

$$\rho\,\overbrace{4\,\pi\,r^2}^{A}\,U_{br} = \rho\,U_p\,A_p = m_i$$

Hence,

$$U_{br} = \frac{m_i}{\rho\,4\,\pi\,r^2}$$

The center velocity is (also) exactly $U_{br}$. The total velocity of boundary is

$$U_t = \frac{m_i}{\rho\,4\,\pi\,r^2}\left(\hat{x} + \hat{r}\right)$$

It can be noticed that the velocity at the opposite to the connection to the rigid pipe which is double of the center velocity.

End Solution

#### 5.3.2.3 One–Dimensional Control Volume

Additional simplification of the continuity equation is of one dimensional flow. This simplification provides very useful description for many fluid flow phenomena. The main assumption made in this model is that the proprieties in the across section are only function of $x$ coordinate . This assumptions leads

$$\int_{A_2} \rho_2\,U_2\,dA - \int_{A_1} \rho_1\,U_1\,dA = \frac{d}{dt}\int_{V(x)} \rho(x)\,\overbrace{A(x)\,dx}^{dV} \tag{5.16}$$

When the density can be considered constant equation (5.16) is reduced to

$$\int_{A_2} U_2\, dA - \int_{A_1} U_1\, dA = \frac{d}{dt}\int A(x)dx \tag{5.17}$$

For steady state but with variations of the velocity and variation of the density reduces equation (5.16) to become

$$\int_{A_2} \rho_2\, U_2\, dA = \int_{A_1} \rho_1\, U_1\, dA \tag{5.18}$$

For steady state and uniform density and velocity equation (5.18) reduces further to

$$\rho_1\, A_1\, U_1 = \rho_2\, A_2\, U_2 \tag{5.19}$$

For incompressible flow (constant density), continuity equation is at its minimum form of

$$U_1\, A_1 = A_2\, U_2 \tag{5.20}$$

The next example is of semi one–dimensional example to illustrate equation (5.16).

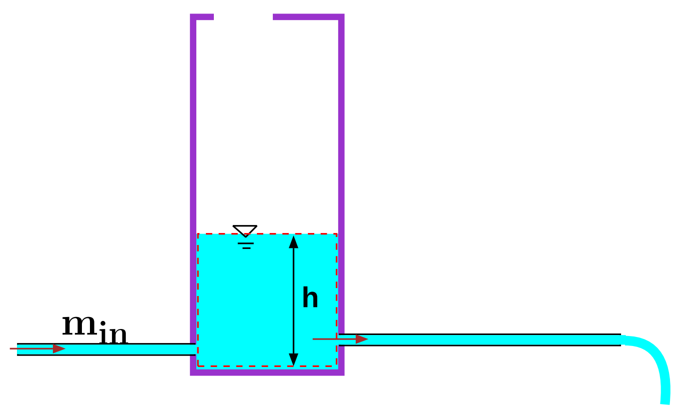



*Fig. -5.6. Height of the liquid for example 5.4.*

Example 5.4:

*Liquid flows into tank in a constant mass flow rate of $a$. The mass flow rate out is function of the height. First assume that $q_{out} = b\, h$ second Assume as $q_{out} = b\,\sqrt{h}$. For the first case, determine the height, $h$ as function of the time. Is there a critical value and then if exist find the critical value of the system parameters. Assume that the height at time zero is $h_0$. What happen if the $h_0 = 0$?*

Solution

The control volume for both cases is the same and it is around the liquid in the tank. It can be noticed that control volume satisfy the demand of one dimensional since the flow is only function of $x$ coordinate. For case one the right hand side term in equation (5.16) is

$$\rho \frac{d}{dt} \int_0^L h\, dx = \rho\, L \frac{d\, h}{dt}$$

Substituting into equation equation (5.16) is

$$\rho\, L\, \frac{d\, h}{d\, t} = \overbrace{b_1\, h}^{\text{flow out}} - \overbrace{m_i}^{\text{flow in}}$$

solution is

$$h = \overbrace{\frac{m_i}{b_1}\, \mathsf{e}^{-\frac{b_1\, t}{\rho\, L}}}^{\text{homogeneous solution}} + \overbrace{c_1\, \mathsf{e}^{\frac{b_1\, t}{\rho\, L}}}^{\text{private solution}}$$

The solution has the homogeneous solution (solution without the $m_i$) and the solution of the $m_i$ part. The solution can rearranged to a new form (a discussion why this form is preferred will be provided in dimensional chapter).

$$\frac{h\, b_1}{m_1} = \mathsf{e}^{-\frac{b_1\, t}{\rho\, L}} + c\, \mathsf{e}^{\frac{b_1\, t}{\rho\, L}}$$

With the initial condition that at $h(t = 0) = h_0$ the constant coefficient can be found as

$$\frac{h_0\, b_1}{m_1} = 1 - c \Longrightarrow c = 1 - \frac{h_0\, b_1}{m_i}$$

which the solution is

$$\frac{h\, b_1}{m_1} = \mathsf{e}^{-\frac{b_1\, t}{\rho\, L}} + \left[1 - \frac{h_0\, b_1}{m_i}\right] \mathsf{e}^{\frac{b_1\, t}{\rho\, L}}$$

It can be observed that if $1 = \frac{h_0\, b_1}{m_i}$ is the critical point of this solution. If the term $\frac{h_0\, b_1}{m_i}$ is larger than one then the solution reduced to a negative number. However, negative number for height is not possible and the height solution approach zero. If the reverse case appeared, the height will increase. Essentially, the critical ratio state if the flow in is larger or lower than the flow out determine the condition of the height.

For second case, the governing equation (5.16) is

$$\rho\, L\, \frac{d\, h}{d\, t} = \overbrace{b\, \sqrt{h}}^{\text{flow out}} - \overbrace{m_i}^{\text{flow in}}$$

with the general solution of

$$\ln\left[\left(\frac{\sqrt{h}\, b}{m_i} - 1\right) \frac{m_i}{\rho\, L}\right] + \frac{\sqrt{h}\, b}{m_i} - 1 = (t + c)\, \frac{\sqrt{h}\, b}{2\, \rho\, L}$$

The constant is obtained when the initial condition that at $h(t=0)=h_0$ and it left as exercise for the reader.

End Solution

## 5.4 Reynolds Transport Theorem

It can be noticed that the same derivations carried for the density can be carried for other intensive properties such as specific entropy, specific enthalpy. Suppose that $g$ is intensive property (which can be a scalar or a vector) undergoes change with time. The change of accumulative property will be then

$$\frac{D}{Dt}\int_{sys} f\,\rho dV = \frac{d}{dt}\int_{c.v.} f\,\rho dV + \int_{c.v} f\,\rho\, U_{rn} dA \tag{5.21}$$

This theorem named after Reynolds, Osborne, (1842-1912) which is actually a three dimensional generalization of Leibniz integral rule[1]. To make the previous derivation clearer, the Reynolds Transport Theorem will be reproofed and discussed. The ideas are the similar but extended some what.

Leibniz integral rule[2] is an one dimensional and it is defined as

$$\frac{d}{dy}\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx = \int_{x_1(y)}^{x_2(y)} \frac{\partial f}{\partial y}\,dx + f(x_2,y)\,\frac{dx_2}{dy} - f(x_1,y)\,\frac{dx_1}{dy} \tag{5.22}$$

Initially, a proof will be provided and the physical meaning will be explained. Assume that there is a function that satisfy the following

$$G(x,y) = \int^{x} f\,(\alpha,\,y)\;d\alpha \tag{5.23}$$

Notice that lower boundary of the integral is missing and is only the upper limit of the function is present[3]. For its derivative of equation (5.23) is

$$f(x,y) = \frac{\partial G}{\partial x} \tag{5.24}$$

differentiating (chain rule $d\,uv = u\,dv + v\,du$) by part of left hand side of the Leibniz integral rule (it can be shown which are identical) is

$$\frac{d\,\left[G(x_2,y) - G(x_1,y)\right]}{dy} = \overbrace{\frac{\partial G}{\partial x_2}\frac{dx_2}{dy}}^{1} + \overbrace{\frac{\partial G}{\partial y}(x_2,y)}^{2} - \overbrace{\frac{\partial G}{\partial x_1}\frac{dx_1}{dy}}^{3} - \overbrace{\frac{\partial G}{\partial y}(x_1,y)}^{4} \tag{5.25}$$

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[1]These papers can be read on-line at http://www.archive.org/details/papersonmechanic01reynrich.

[2]This material is not necessarily but is added her for completeness. This author find material just given so no questions will be asked.

[3]There was a suggestion to insert arbitrary constant which will be canceled and will a provide rigorous proof. This is engineering book and thus, the exact mathematical proof is not the concern here. Nevertheless, if there will be a demand for such, it will be provided.

The terms 2 and 4 in equation (5.25) are actually (the $x_2$ is treated as a different variable)

$$\int_{x_1(y)}^{x_2(y)} \frac{\partial\, f(x,y)}{\partial y}\, dx \tag{5.26}$$

The first term (1) in equation (5.25) is

$$\frac{\partial G}{\partial x_2}\frac{dx_2}{dy} = f(x_2,y)\,\frac{dx_2}{dy} \tag{5.27}$$

The same can be said for the third term (3). Thus this explanation is a proof the Leibniz rule.

The above "proof" is mathematical in nature and physical explanation is also provided. Suppose that a fluid is flowing in a conduit. The intensive property, $f$ is investigated or the accumulative property, $F$. The interesting information that commonly needed is the change of the accumulative property, $F$, with time. The change with time is

$$\frac{DF}{Dt} = \frac{D}{Dt}\int_{sys} \rho\, f\, dV \tag{5.28}$$

For one dimensional situation the change with time is

$$\frac{DF}{Dt} = \frac{D}{Dt}\int_{sys} \rho\, f\, A(x) dx \tag{5.29}$$

If two limiting points (for the one dimensional) are moving with a different coordinate system, the mass will be different and it will not be a system. This limiting condition is the control volume for which some of the mass will leave or enter. Since the change is very short (differential), the flow in (or out) will be the velocity of fluid minus the boundary at $x_1$, $U_{rn} = U_1 - U_b$. The same can be said for the other side. The accumulative flow of the property in, $F$, is then

$$F_{in} = \overbrace{f_1\,\rho}^{F_1}\,\overbrace{U_{rn}}^{\frac{dx_1}{dt}} \tag{5.30}$$

The accumulative flow of the property out, $F$, is then

$$F_{out} = \overbrace{f_2\,\rho}^{F_2}\,\overbrace{U_{rn}}^{\frac{dx_2}{dt}} \tag{5.31}$$

The change with time of the accumulative property, $F$, between the boundaries is

$$\frac{d}{dt}\int_{c.v.} \rho(x)\, f\, A(x)\, dA \tag{5.32}$$

When put together it brings back the Leibniz integral rule. Since the time variable, $t$, is arbitrary and it can be replaced by any letter. The above discussion is one of the physical meaning the Leibniz rule.

Reynolds Transport theorem is a generalization of the Leibniz rule and thus the same arguments are used. The only difference is that the velocity has three components and only the perpendicular component enters into the calculations.

Reynolds Transport

$$\frac{D}{DT}\int_{sys} f\,\rho dV = \frac{d}{dt}\int_{c.v} f\,\rho\, dV + \int_{S_{c.v.}} f\,\rho\, U_{rn}\, dA \tag{5.33}$$

## 5.5 Examples For Mass Conservation

Several examples are provided to illustrate the topic.

Example 5.5:

*Liquid enters a circular pipe with a linear velocity profile as a function of the radius with maximum velocity of $U_{max}$. After magical mixing, the velocity became uniform. Write the equation which describes the velocity at the entrance. What is the magical averaged velocity at the exit? Assume no–slip condition.*

SOLUTION

The velocity profile is linear with radius. Additionally, later a discussion on relationship between velocity at interface to solid also referred as the (no) slip condition will be provided. This assumption is good for most cases with very few exceptions. It will be assumed that the velocity at the interface is zero. Thus, the boundary condition is $U(r=R)=0$ and $U(r=0)=U_{max}$ Therefore the velocity profile is

$$U(r) = U_{max}\left(1-\frac{r}{R}\right)$$

Where $R$ is radius and $r$ is the working radius (for the integration). The magical averaged velocity is obtained using the equation (5.13). For which

$$\int_0^R U_{max}\left(1-\frac{r}{R}\right)\, 2\,\pi\, r\, dr = U_{ave}\,\pi\, R^2 \tag{5.V.a}$$

The integration of the equation (5.V.a) is

$$U_{max}\,\pi\,\frac{R^2}{6} = U_{ave}\,\pi\, R^2 \tag{5.V.b}$$

The solution of equation (b) results in average velocity as

$$U_{ave} = \frac{U_{max}}{6} \tag{5.V.c}$$

End Solution

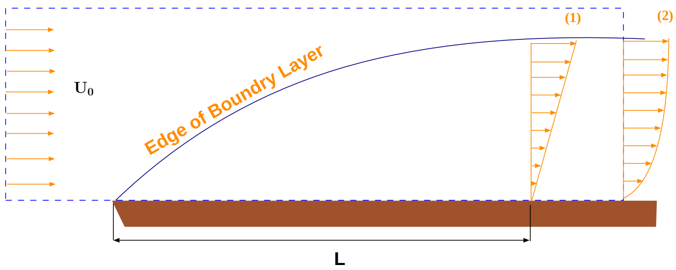



Fig. -5.7. Boundary Layer control mass.

Example 5.6:

*Experiments have shown that a layer of liquid that attached itself to the surface and it is referred to as boundary layer. The assumption is that fluid attaches itself to surface. The slowed liquid is slowing the layer above it. The boundary layer is growing with $x$ because the boundary effect is penetrating further into fluid. A common boundary layer analysis uses the Reynolds transform theorem. In this case, calculate the relationship of the mass transfer across the control volume. For simplicity assume slowed fluid has a linear velocity profile. Then assume parabolic velocity profile as*

$$U_x(y) = 2\,U_0 \left[\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right]$$

*and calculate the mass transfer across the control volume. Compare the two different velocity profiles affecting on the mass transfer.*

Solution

Assuming the velocity profile is linear thus, (to satisfy the boundary condition) it will be

$$U_x(y) = \frac{U_0\,y}{\delta}$$

The chosen control volume is rectangular of $L \times \delta$. Where $\delta$ is the height of the boundary layer at exit point of the flow as shown in Figure 5.7. The control volume has three surfaces that mass can cross, the left, right, and upper. No mass can cross the lower surface (solid boundary). The situation is steady state and thus using equation (5.13) results in

$$\overbrace{\overbrace{\int_0^\delta U_0\,dy}^{in} - \overbrace{\int_0^\delta \frac{U_0\,y}{\delta}\,dy}^{out}}^{\text{x direction}} = \overbrace{\int_0^L U x dx}^{\text{y direction}}$$

It can be noticed that the convention used in this chapter of "in" as negative is not "followed." The integral simply multiply by negative one. The above integrals on the right hand side can be combined as

$$\int_0^{\delta} U_0 \left(1 - \frac{y}{\delta}\right)\, dy = \int_0^L U x dx$$

the integration results in

$$\frac{U_0\, \delta}{2} = \int_0^L U x dx$$

or for parabolic profile

$$\int_0^{\delta} U_0\, dy - \int_0^{\delta} U_0 \left[\frac{y}{\delta} + \left(\frac{y}{\delta}\right)^2\right] dy = \int_0^L U x dx$$

or

$$\int_0^{\delta} U_0 \left[1 - \frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right] dy = U_0$$

the integration results in

$$\frac{U_0\, \delta}{2} = \int_0^L U x dx$$

End Solution

Example 5.7:

*Air flows into a jet engine at $5\, kg/sec$ while fuel flow into the jet is at $0.1\, kg/sec$. The burned gases leaves at the exhaust which has cross area $0.1\, m^2$ with velocity of $500\, m/sec$. What is the density of the gases at the exhaust?*

SOLUTION

The mass conservation equation (5.13) is used. Thus, the flow out is ( 5 + 0.1 ) $5.1\, kg/sec$ The density is

$$\rho = \frac{\dot{m}}{A\, U} = \frac{5.1\, kg/sec}{0.01\, m^2\; 500\, m/sec} = 1.02 kg/m^3$$

End Solution

The mass (volume) flow rate is given by direct quantity like $x\, kg/sec$. However sometime, the mass (or the volume) is given by indirect quantity such as the effect of flow. The next example deal with such reversed mass flow rate.

Example 5.8:

*The tank is filled by two valves which one filled tank in 3 hours and the second by 6 hours. The tank also has three emptying valves of 5 hours, 7 hours, and 8 hours. The tank is 3/4 fulls, calculate the time for tank reach empty or full state when all the valves are open. Is there a combination of valves that make the tank at steady state?*

Solution

Easier measurement of valve flow rate can be expressed as fraction of the tank per hour. For example valve of 3 hours can be converted to 1/3 tank per hour. Thus, mass flow rate in is

$$\dot{m}_{in} = 1/3 + 1/6 = 1/2 tank/hour$$

The mass flow rate out is

$$\dot{m}_{out} = 1/5 + 1/7 + 1/8 = \frac{131}{280}$$

Thus, if all the valves are open the tank will be filled. The time to completely filled the tank is

$$\frac{\frac{1}{4}}{\frac{1}{2} - \frac{131}{280}} = \frac{70}{159} hour$$

The rest is under construction.

End Solution

Example 5.9:

*Inflated cylinder is supplied in its center with constant mass flow. Assume that the gas mass is supplied in uniformed way of $m_i\,[kg/m/sec]$. Assume that the cylinder inflated uniformly and pressure inside the cylinder is uniform. The gas inside the cylinder obeys the ideal gas law. The pressure inside the cylinder is linearly proportional to the volume. For simplicity, assume that the process is isothermal. Calculate the cylinder boundaries velocity.*

Solution

The applicable equation is

$$\overbrace{\int_{V_{c.v}} \frac{d\rho}{dt}\, dV}^{\text{increase pressure}} + \overbrace{\int_{S_{c.v.}} \rho\, U_b dV}^{\text{boundary velocity}} = \overbrace{\int_{S_{c.v.}} \rho U_{rn}\, dA}^{\text{in or out flow rate}}$$

Every term in the above equation is analyzed but first the equation of state and volume to pressure relationship have to be provided.

$$\rho = \frac{P}{R\,T}$$

and relationship between the volume and pressure is

$$P = f\, \pi\, {R_c}^2$$

Where $R_c$ is the instantaneous cylinder radius. Combining the above two equations results in

$$\rho = \frac{f\,\pi\,{R_c}^2}{R\,T}$$

Where $f$ is a coefficient with the right dimension. It also can be noticed that boundary velocity is related to the radius in the following form

$$U_b = \frac{dR_c}{dt}$$

The first term requires to find the derivative of density with respect to time which is

$$\frac{d\rho}{dt} = \frac{d}{dt}\left(\frac{f\,\pi\,{R_c}^2}{R\,T}\right) = \frac{2\,f\,\pi\,R_c}{R\,T}\,\frac{\overbrace{dR_c}^{U_b}}{dt}$$

Thus the first term is

$$\int_{V_{c.v}} \frac{d\rho}{dt}\,\overbrace{dV}^{2\,\pi\,R_c} = \int_{V_{c.v}} \frac{2\,f\,\pi\,R_c}{R\,T}U_b\,\overbrace{dV}^{2\,\pi\,R_c\,dR_c} = \frac{4\,f\,\pi^2\,{R_c}^3}{3\,R\,T}U_b$$

The integral can be carried when $U_b$ is independent of the $R_c$[4] The second term is

$$\int_{S_{c.v.}} \rho\,U_b dA = \overbrace{\frac{f\,\pi\,{R_c}^2}{R\,T}}^{\rho}\,U_b\,\overbrace{2\,\pi R_c}^{A} = \left(\frac{f\,\pi^3\,{R_c}^2}{R\,T}\right)U_b$$

substituting in the governing equation obtained the form of

$$\frac{f\,\pi^2\,{R_c}^3}{R\,T}U_b + \frac{4\,f\,\pi^2\,{R_c}^3}{3\,R\,T}\,U_b = m_i$$

The boundary velocity is then

$$U_b = \frac{m_i}{\dfrac{7\,f\,\pi^2\,{R_c}^3}{3\,R\,T}}G = \frac{3\,m_i\,R\,T}{7\,f\,\pi^2\,{R_c}^3}$$

End Solution

### Example 5.10:

*A balloon is attached to a rigid supply and is supplied by a constant mass rate, $m_i$. Assume that gas obeys the ideal gas law. Assume that balloon volume is a linear function of the pressure inside the balloon such as $P = f_v\,V$. Where $f_v$ is a coefficient describing the balloon physical characters. Calculate the velocity of the balloon boundaries under the assumption of isothermal process.*

[4] The proof of this idea is based on the chain differentiation similar to Leibniz rule. When the derivative of the second part is $dU_b/dR_c = 0$.

Solution

The question is more complicated than Example 5.10. The ideal gas law is

$$\rho = \frac{P}{R\,T}$$

The relationship between the pressure and volume is

$$P = f_v\,V = \frac{4\,f_v\,\pi\,{R_b}^3}{3}$$

The combining of the ideal gas law with the relationship between the pressure and volume results

$$\rho = \frac{4\,f_v\,\pi\,{R_b}^3}{3\,R\,T}$$

The applicable equation is

$$\int_{V_{c.v}} \frac{d\rho}{dt}\,dV + \int_{S_{c.v.}} \rho\,\left(U_c\,\hat{x} + U_b\hat{r}\right) dA = \int_{S_{c.v.}} \rho U_{rn}\,dA$$

The right hand side of the above equation is

$$\int_{S_{c.v.}} \rho U_{rn}\,dA = m_i$$

The density change is

$$\frac{d\rho}{dt} = \frac{12\,f_v\,\pi\,{R_b}^2}{R\,T}\,\overbrace{\frac{dR_b}{dt}}^{U_b}$$

The first term is

$$\int_0^{R_b} \overbrace{\frac{12\,f_v\,\pi\,{R_b}^2}{R\,T}\,U_b}^{\neq f(r)}\,\overbrace{4\,\pi\,r^2\,dr}^{dV} = \frac{16\,f_v\,\pi^2\,{R_b}^5}{3\,R\,T}\,U_b$$

The second term is

$$\int_A \frac{4\,f_v\,\pi\,{R_b}^3}{3\,R\,T}\,U_b\,dA = \frac{4\,f_v\,\pi\,{R_b}^3}{3\,R\,T}\,U_b\,\overbrace{4\,\pi\,{R_b}^2}^{A} = \frac{8\,f_v\,\pi^2\,{R_b}^5}{3\,R\,T}\,U_b$$

Subsisting the two equations of the applicable equation results

$$U_b = \frac{1}{8}\,\frac{m_i\,R\,T}{f_v\,\pi^2\,{R_b}^5}$$

Notice that first term is used to increase the pressure and second the change of the boundary.

End Solution

**Open Question: Answer must be received by April 15, 2010**

The best solution of the following question will win 18 U.S. dollars and your name will be associated with the solution in this book.

Example 5.11:
*Solve example 5.10 under the assumption that the process is isentropic. Also assume that the relationship between the pressure and the volume is $P = f_v V^2$. What are the units of the coefficient $f_v$ in this problem? What are the units of the coefficient in the previous problem?*

## 5.6 The Details Picture – Velocity Area Relationship

The integral approach is intended to deal with the "big" picture. Indeed the method is used in this part of the book for this purpose. However, there is very little written about the usability of this approach to provide way to calculate the average quantities in the control system. Sometimes it is desirable to find the averaged velocity or velocity distribution inside a control volume. There is no general way to provide these quantities. Therefore an example will be provided to demonstrate the use of this approach.

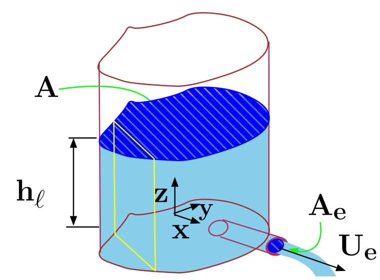



*Fig. -5.8. Control volume usage to calculate local averaged velocity in three coordinates.*

Consider a container filled with liquid on which one exit opened and the liquid flows out as shown in Figure 5.8. The velocity has three components in each of the coordinates under the assumption that flow is uniform and the surface is straight[5]. The integral approached is used to calculate the averaged velocity of each to the components. To relate the velocity in the $z$ direction with the flow rate out or the exit the velocity mass balance is constructed. A similar control volume construction to find the velocity of the boundary velocity (height) can be carried out. The control volume is bounded by the container wall including the exit of the flow. The upper boundary is surface parallel to upper surface but at $Z$ distance from the bottom. The mass balance reads

$$\int_V \frac{d\rho}{dt}\, dV + \int_A U_{bn}\, \rho\, dA + \int_A U_{rn}\, \rho\, dA = 0 \tag{5.34}$$

[5]The liquid surface is not straight for this kind of problem. However, under certain conditions it is reasonable to assume straight surface which have been done for this problem.

For constant density (conservation of volume) equation[6] and $(h > z)$ reduces to

$$\int_A U_{rn}\, \rho\, dA = 0 \tag{5.35}$$

In the container case for uniform velocity equation 5.35 becomes

$$U_z\, A = U_e\, A_e \Longrightarrow U_z = -\frac{A_e}{A} U_e \tag{5.36}$$

It can be noticed that the boundary is not moving and the mass inside does not change this control volume. The velocity $U_z$ is the averaged velocity downward.

The $x$ component of velocity is obtained by using a different control volume. The control volume is shown in Figure 5.9. The boundary are the container far from the flow exit with blue line projection into page (area) shown in the Figure 5.9. The mass conservation for constant density of this control volume is

$$-\int_A U_{bn}\, \rho\, dA + \int_A U_{rn}\, \rho\, dA = 0 \tag{5.37}$$

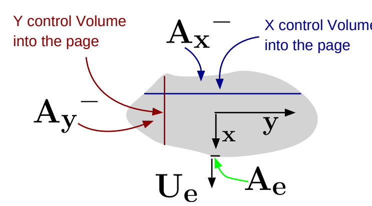



*Fig. -5.9. Control volume and system before and after the motion.*

Usage of control volume not included in the previous analysis provides the velocity at the upper boundary which is the same as the velocity at $y$ direction. Substituting into (5.37) results in

$$\int_{A_x{}^-} \frac{A_e}{A} U_e\, \rho\, dA + \int_{A_{yz}} U_x\, \rho\, dA = 0 \tag{5.38}$$

Where $A_x{}^-$ is the area shown the Figure under this label. The area $A_{yz}$ referred to area into the page in Figure 5.9 under the blow line. Because averaged velocities and constant density are used transformed equation (5.38) into

$$\frac{A_e}{A} A_x{}^- U_e + U_x \overbrace{Y(x)\, h}^{A_{yz}} = 0 \tag{5.39}$$

Where $Y(x)$ is the length of the (blue) line of the boundary. It can be notice that the velocity, $U_x$ is generally increasing with $x$ because $A_x{}^-$ increase with $x$.

The calculations for the $y$ directions are similar to the one done for $x$ direction. The only difference is that the velocity has two different directions. One zone is right to the exit with flow to the left and one zone to left with averaged velocity to right. If the volumes on the left and the right are symmetrical the averaged velocity will be zero.

[6]The point where $(z = h)$ the boundary term is substituted the flow in term.

Example 5.12:
*Calculate the velocity, $U_x$ for a cross section of circular shape (cylinder).*

Solution

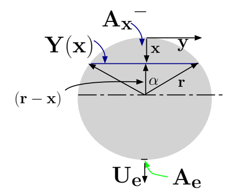



*Fig. -5.10. Circular cross section for finding $U_x$ and various cross sections.*

The relationship for this geometry needed to be expressed. The length of the line $Y(x)$ is

$$Y(x) = 2\,r\,\sqrt{1 - \left(1 - \frac{x}{r}\right)^2} \tag{5.XII.a}$$

This relationship also can be expressed in the term of $\alpha$ as

$$Y(x) = 2\,r\,\sin\alpha \tag{5.XII.b}$$

Since this expression is simpler it will be adapted. When the relationship between radius angle and $x$ are

$$x = r(1 - \sin\alpha) \tag{5.XII.c}$$

The area $A_x{}^-$ is expressed in term of $\alpha$ as

$$A_x{}^- = \left(\alpha - \frac{1}{2}, \sin(2\alpha)\right) r^2 \tag{5.XII.d}$$

Thus the velocity, $U_x$ is

$$\frac{A_e}{A}\left(\alpha - \frac{1}{2}\,\sin(2\alpha)\right) r^2\, U_e + U_x\, 2\,r\,\sin\alpha\, h = 0 \tag{5.XII.e}$$

$$U_x = \frac{A_e}{A}\,\frac{r}{h}\,\frac{\left(\alpha - \frac{1}{2}\,\sin(2\alpha)\right)}{\sin\alpha}\,U_e \tag{5.XII.f}$$

Averaged velocity is defined as

$$\overline{U_x} = \frac{1}{S}\int_S U dS \tag{5.XII.g}$$

Where here $S$ represent some length. The same way it can be represented for angle calculations. The value $dS$ is $r\cos\alpha$. Integrating the velocity for the entire container and dividing by the angle, $\alpha$ provides the averaged velocity.

$$\overline{U_x} = \frac{1}{2\,r}\int_0^\pi \frac{A_e}{A}\,\frac{r}{h}\,\frac{\left(\alpha - \frac{1}{2}\,\sin(2\alpha)\right)}{\tan\alpha}\,U_e\,r\,d\alpha \tag{5.XII.h}$$

which results in

$$\overline{U_x} = \frac{(\pi - 1)}{4}\,\frac{A_e}{A}\,\frac{r}{h}\,U_e \tag{5.XII.i}$$

End Solution

Example 5.13:

*Calculate the velocity, $U_y$ for a cross section of circular shape (cylinder). What is the averaged velocity if only half section is used. State your assumptions and how it similar to the previous example.*

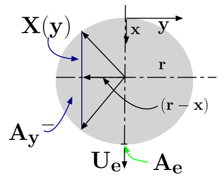



*Fig. -5.11. $y$ velocity for a circular shape*

Solution

The flow out in the $x$ direction is zero because symmetrical reasons. That is the flow field is a mirror images. Thus, every point has different velocity with the same value in the opposite direction.

The flow in half of the cylinder either the right or the left has non zero averaged velocity. The calculations are similar to those in the previous to example 5.12. The main concept that must be recognized is the half of the flow must have come from one side and the other come from the other side. Thus, equation (5.39) modified to be

$$\frac{A_e}{A} A_x{}^{-} U_e + U_x \overbrace{Y(x)\, h}^{A_{yz}} = 0 \tag{5.40}$$

The integral is the same as before but the upper limit is only to $\pi/2$

$$\overline{U_x} = \frac{1}{2\,r} \int_0^{\pi/2} \frac{A_e}{A} \frac{r}{h} \frac{\left(\alpha - \frac{1}{2}\sin(2\alpha)\right)}{\tan\alpha} U_e\, r\, d\alpha \tag{5.XIII.a}$$

which results in

$$\overline{U_x} = \frac{(\pi - 2)}{8} \frac{A_e}{A} \frac{r}{h} U_e \tag{5.XIII.b}$$

End Solution

## 5.7 More Examples for Mass Conservation

Typical question about the relative velocity that appeared in many fluid mechanics exams is the following.

Example 5.14:

*A boat travels at speed of 10$m/sec$ upstream in a river that flows at a speed of 5$m/s$. The inboard engine uses a pump to suck in water at the front $A_{in} = 0.2\,m^2$ and eject it through the back of the boat with exist area of $A_{out} = 0.05\,m^2$. The water absolute velocity leaving the back is 50$m/sec$, what are the relative velocities entering and leaving the boat and the pumping rate?*

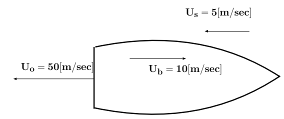



*Fig. -5.12. Schematic of the boat for example 5.14*

Solution

The boat is assumed (implicitly is stated) to be steady state and the density is constant. However, the calculation have to be made in the frame of reference moving with the boat. The relative jet discharge velocity is

$$U_{r_{out}} = 50 - (10 + 5) = 35[m/sec]$$

The volume flow rate is then

$$Q_{out} = A_{out}\, U_{r_{out}} = 35 \times 0.05 = 1.75 m^3/sec$$

The flow rate at entrance is the same as the exit thus,

$$U_{r_{in}} = \frac{A_{out}}{A_{in}}\, U_{r_{out}} = \frac{0.05}{0.2} 35 = 8.75 m/sec$$

End Solution

Example 5.15:

*Liquid A enters a mixing device depicted in at 0.1 [$kg/s$]. In same time liquid B enter the mixing device with a different specific density at 0.05 [$kg/s$]. The density of liquid A is 1000[$kg/m^3$] and liquid B is 800[$kg/m^3$]. The results of the mixing is a homogeneous mixture. Assume incompressible process. Find the average leaving velocity and density of the mixture leaving through the 2O [$cm$] diameter pipe. If the mixing device volume is decreasing (as a piston pushing into the chamber) at rate of .002 [$m^3/s$], what is the exit velocity? State your assumptions.*

Solution

In the first scenario, the flow is steady state and equation (5.11) is applicable

$$\dot{m}_A + \dot{m}_B = Q_{mix}\, \rho_{mix} \Longrightarrow= 0.1 + 0.05 = 0.15[m] \tag{5.XV.a}$$

Thus in this case, since the flow is incompressible flow, the total volume flow in is equal to volume flow out as

$$\dot{Q}_A + \dot{Q}_B = \dot{Q}_{mix} \Longrightarrow= \frac{\dot{m}_A}{\rho_A} + \frac{\dot{m}_A}{\rho_A} = \frac{0.10}{1000} + \frac{0.05}{800}$$

Thus the mixture density is

$$\rho_{mix} = \frac{\dot{m}_A + \dot{m}_B}{\dfrac{\dot{m}_A}{\rho_A} + \dfrac{\dot{m}_B}{\rho_B}} = 923.07[kg/m^3] \tag{5.XV.b}$$

The averaged velocity is then

$$U_{mix} = \frac{Q_{mix}}{A_{out}} = \frac{\dfrac{\dot{m}_A}{\rho_A} + \dfrac{\dot{m}_B}{\rho_B}}{\pi\, 0.01^2} = \frac{1.625}{\pi}[m/s] \tag{5.XV.c}$$

In the case that a piston is pushing the exit density could be changed and fluctuated depending on the location of the piston. However, if the assumption of well mixed is still holding the exit density should not affected. The term that should be added to the governing equation the change of the volume. So governing equation is (5.15).

$$\overbrace{U_{bn}\, A\, \rho_b}^{-Q_b\, \rho_{mix}} = \overbrace{\dot{m}_A + \dot{m}_B}^{in} - \overbrace{\dot{m}_{mix}}^{out} \tag{5.XV.d}$$

That is the mixture device is with an uniform density

$$-0.002[m/sec]\, 923.7[kg/m^3] = 0.1 + 0.05 - m_{exit} \tag{5.XV.e}$$

$$m_{exit} = 1.9974[kg/s]$$

End Solution

Example 5.16:

*A syringe apparatus is being use to withdrawn blood[7]. If the piston is withdrawn at O.01 $[m/s]$. At that stage air leaks in around the piston at the rate 0.000001 $[m^3/s]$. What is the average velocity of blood into syringe (at the tip)? The syringe radios is 0.005[m] and the tip radius is 0.0003 [m].*

Solution

The situation is unsteady state (in the instinctive c.v. and coordinates) since the mass in the control volume (the syringe volume is not constant). The chose of the control volume and coordinate system determine the amount of work. This part of the solution is art. There are several possible control volumes that can be used to solve the problem. The two "instinctive control volumes" are the blood with the air and the the whole volume between the tip and syringe plunger (piston). The first choice seem reasonable

[7]The author still remember his elementary teacher that was so appalled by the discussion on blood piping which students in an engineering school were doing. He gave a speech about how inhuman these engineering students are. I hope that no one will have teachers like him. Yet, it can be observed that bioengineering is "cool" today while in 40 years ago is a disgusting field.

since it provides relationship of the total to specific material. In that case, control volume is the volume syringe tip to the edge of the blood. The second part of the control volume is the air. For this case, the equation (5.15) is applicable and can be written as

$$U_{tip}\,A_{tip}\,\cancel{\rho_b} = U_b\,A_s\,\cancel{\rho_b} \tag{5.XVI.a}$$

In the air side the same equation can used. There several coordinate systems that can used, attached to plunger, attached to the blood edge, stationary. Notice that change of the volume do not enter into the calculations because the density of the air is assumed to be constant. In stationary coordinates two boundaries are moving and thus

$$\overbrace{U_{plunger}\,A_s\,\rho_a - U_b\,A_s\,\rho_b}^{\text{moving b.c.}} = \overbrace{\rho_a \dot{Q}_{in}}^{\text{in/out}} \tag{5.XVI.b}$$

In the case, the choice is coordinates moving with the plunger, the relative plunger velocity is zero while the blood edge boundary velocity is $U_{plunger} - U_b$. The air governing equation is

$$\overbrace{\left(U_{plunger} - U_b\right)}^{\text{blood b. velocity}}\,A_s\,\rho_b = \overbrace{\rho_a \dot{Q}_{in}}^{\text{in/out}} \tag{5.XVI.c}$$

In the case of coordinates are attached to the blood edge similar equation is obtained. At this stage, there are two unknowns, $U_b$ and $U_{tip}$, and two equations. Using equations (5.XVI.a) and (5.XVI.c) results in

$$\begin{gathered} U_b = U_{plunger} - \frac{\rho_a\,Q_{in}}{A_s\,\rho_b} \\ U_{tip} = \frac{U_b\,A_s}{A_{tip}} = \frac{\left(U_{plunger} - \dfrac{\rho_a\,Q_{in}}{A_s\,\rho_b}\right)A_s}{A_{tip}} \end{gathered} \tag{5.XVI.d}$$

End Solution

Example 5.17:

*The apparatus depicted in Figure* **??** *is referred in the literature sometime as the water–jet pump. In this device, the water (or another liquid) is pumped throw the inner pipe at high velocity. The outside pipe is lower pressure which suck the water (other liquid) into device. Later the two stream are mixed. In this question the what is the mixed stream averaged velocity with* $U_1 = 4.0[m/s]$ *and* $U_2 = 0.5[m/s]$. *The cross section inside and outside radii ratio is* $r_1/r_2 = 0.2$. *Calculate the mixing averaged velocity.*

Solution

The situation is steady state and which density of the liquid is irrelevant (because it is the same at the inside and outside).

$$U_1\,A_1 + U_2\,A_2 = U_3\,A_3 \tag{5.XVII.a}$$

The velocity is $A_3 = A_1 + A_2$ and thus

$$U_3 = \frac{U_1\,A_1 + U_2\,A_2}{A_3} == U_1\frac{A_1}{A_3} + U_2\left(1 - \frac{A_1}{A_3}\right) \tag{5.XVII.b}$$

End Solution

# CHAPTER 6

# Momentum Conservation for Control Volume

## 6.1 Momentum Governing Equation

### 6.1.1 Introduction to Continuous

In the previous chapter, the Reynolds Transport Theorem (RTT) was applied to mass conservation. Mass is a scalar (quantity without magnitude). This chapter deals with momentum conservation which is a vector. The Reynolds Transport Theorem (RTT) is applicable to any quantity and the discussion here will deal with forces that acting on the control volume. Newton's second law for single body is as the following

$$\boldsymbol{F} = \frac{d(m\boldsymbol{U})}{dt} \tag{6.1}$$

It can be noticed that bold notation for the velocity is $\boldsymbol{U}$ (and not $U$) to represent that the velocity has a direction. For several bodies ($n$), Newton's law becomes

$$\sum_{i=1}^{n} \boldsymbol{F}_i = \sum_{i=1}^{n} \frac{d(m\boldsymbol{U})_i}{dt} \tag{6.2}$$

The fluid can be broken into infinitesimal elements which turn the above equation (6.2) into a continuous form of small bodies which results in

$$\sum_{i=1}^{n} \boldsymbol{F}_i = \frac{D}{Dt} \int_{sys} \boldsymbol{U} \overbrace{\rho\, dV}^{\text{element mass}} \tag{6.3}$$

Note that the notation $D/Dt$ is used and not $d/dt$ to signify that it referred to a derivative of the system. The Reynold's Transport Theorem (RTT) has to be used on the right hand side.

### 6.1.2 External Forces

First, the terms on the left hand side, or the forces, have to be discussed. The forces, excluding the external forces, are the body forces, and the surface forces as the following

$$\boldsymbol{F}_{total} = \boldsymbol{F}_b + \boldsymbol{F}_s \tag{6.4}$$

In this book (at least in this discussion), the main body force is the gravity. The gravity acts on all the system elements. The total gravity force is

$$\sum \boldsymbol{F}_b = \int_{sys} \boldsymbol{g} \overbrace{\rho\, dV}^{\substack{element\\ mass}} \tag{6.5}$$

which acts through the mass center towards the center of earth. After infinitesimal time the gravity force acting on the system is the same for control volume, hence,

$$\int_{sys} \boldsymbol{g}\, \rho\, dV = \int_{cv} \boldsymbol{g}\, \rho\, dV \tag{6.6}$$

The integral yields a force trough the center mass which has to be found separately.

In this chapter, the surface forces are divided into two categories: one perpendicular to the surface and one with the surface direction (in the surface plain see Figure 6.1.). Thus, it can be written as

$$\sum \boldsymbol{F}_s = \int_{c.v.} \boldsymbol{S_n}\, dA + \int_{c.v.} \boldsymbol{\tau}\, dA \tag{6.7}$$

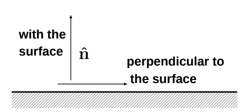



*Fig. -6.1. The explaination for the direction relative to surface perpendicular and with the surface.*

Where the surface "force", $\boldsymbol{S_n}$, is in the surface direction, and $\boldsymbol{\tau}$ are the shear stresses. The surface "force", $\boldsymbol{S_n}$, is made out of two components, one due to viscosity (solid body) and two consequence of the fluid pressure. Here for simplicity, only the pressure component is used which is reasonable for most situations. Thus,

$$\boldsymbol{S}_n = -\boldsymbol{P}\,\hat{n} + \overbrace{\boldsymbol{S_\nu}}^{\sim 0} \tag{6.8}$$

Where $\boldsymbol{S_\nu}$ is perpendicular stress due to viscosity. Again, $\hat{n}$ is an unit vector outward of element area and the negative sign is applied so that the resulting force acts on the body.

### 6.1.3 Momentum Governing Equation

The right hand side, according Reynolds Transport Theorem (RTT), is

$$\frac{D}{Dt}\int_{sys}\rho\boldsymbol{U}dV = \frac{t}{dt}\int_{c.v.}\rho\boldsymbol{U}dV + \int_{c.v.}\rho\boldsymbol{U}\boldsymbol{U}_{rn}dA \tag{6.9}$$

The liquid velocity, $\boldsymbol{U}$, is measured in the frame of reference and $\boldsymbol{U}_{rn}$ is the liquid relative velocity to boundary of the control volume measured in the same frame of reference.

Thus, the general form of the momentum equation without the external forces is

Integral Momentum Equation

$$\begin{aligned}\int_{c.v.}\boldsymbol{g}\,\rho\,dV - \int_{c.v.}\boldsymbol{P}\,dA + \int_{c.v.}\boldsymbol{\tau}\cdot\boldsymbol{dA} \\ = \frac{t}{dt}\int_{c.v.}\rho\boldsymbol{U}dV + \int_{c.v.}\rho\boldsymbol{U}\,\boldsymbol{U_{rn}}\,dV\end{aligned} \tag{6.10}$$

With external forces equation (6.10) is transformed to

Integral Momentum Equation & External Forces

$$\begin{aligned}\sum\boldsymbol{F}_{ext} + \int_{c.v.}\boldsymbol{g}\,\rho\,dV - \int_{c.v.}\boldsymbol{P}\cdot\boldsymbol{dA} + \int_{c.v.}\boldsymbol{\tau}\cdot\boldsymbol{dA} = \\ \frac{t}{dt}\int_{c.v.}\rho\boldsymbol{U}dV + \int_{c.v.}\rho\boldsymbol{U}\,\boldsymbol{U_{rn}}dV\end{aligned} \tag{6.11}$$

The external forces, $F_{ext}$, are the forces resulting from support of the control volume by non–fluid elements. These external forces are commonly associated with pipe, ducts, supporting solid structures, friction (non-fluid), etc.

Equation (6.11) is a vector equation which can be broken into its three components. In Cartesian coordinate, for example in the $x$ coordinate, the components are

$$\begin{aligned}\sum F_x + \int_{c.v.}\left(\boldsymbol{g}\cdot\hat{i}\right)\rho\,dV\int_{c.v.}\boldsymbol{P}\cos\theta_x\,dA + \int_{c.v.}\boldsymbol{\tau}_x\cdot\boldsymbol{dA} = \\ \frac{t}{dt}\int_{c.v.}\rho\boldsymbol{U}_x\,dV + \int_{c.v.}\rho\boldsymbol{U}_x\cdot\boldsymbol{U}_{rn}dA\end{aligned} \tag{6.12}$$

where $\theta_x$ is the angle between $\hat{n}$ and $\hat{i}$ or $(\hat{n}\cdot\hat{i})$.

### 6.1.4 Momentum Equation in Acceleration System

For accelerate system, the right hand side has to include the following acceleration

$$\boldsymbol{a}_{acc} = \boldsymbol{\omega}\times(\boldsymbol{r}\times\boldsymbol{\omega}) + 2\,\boldsymbol{U}\times\boldsymbol{\omega} + \boldsymbol{r}\times\dot{\boldsymbol{\omega}} - \boldsymbol{a}_0 \tag{6.13}$$

Where $\boldsymbol{r}$ is the distance from the center of the frame of reference and the add force is

$$\boldsymbol{F}_{add} = \int_{V_{c.v.}} \boldsymbol{a}_{acc}\, \rho\, dV \tag{6.14}$$

**Integral of Uniform Pressure on Body**

In this kind of calculations, it common to obtain a situation where one of the term will be an integral of the pressure over the body surface. This situation is a similar idea that was shown in Section 4.6. In this case the resulting force due to the pressure is zero to all directions.

### 6.1.5 Momentum For Steady State and Uniform Flow

The momentum equation can be simplified for the steady state condition as it was shown in example 6.3. The unsteady term (where the time derivative) is zero.

Integral Steady State Momentum Equation

$$\sum \boldsymbol{F}_{ext} + \int_{c.v.} \boldsymbol{g}\, \rho\, dV - \int_{c.v.} \boldsymbol{P}\, dA + \int_{c.v.} \boldsymbol{\tau}\, dA = \int_{c.v.} \rho \boldsymbol{U} U_{rn} dA \tag{6.15}$$

#### 6.1.5.1 Momentum for For Constant Pressure and Frictionless Flow

Another important sub category of simplification deals with flow under approximation of the frictionless flow and uniform pressure. This kind of situations arise when friction (forces) is small compared to kinetic momentum change. Additionally, in these situations, flow is exposed to the atmosphere and thus (almost) uniform pressure surrounding the control volume. In this situation, the mass flow rate in and out are equal. Thus, equation (6.15) is further reduced to

$$\boldsymbol{F} = \int_{out} \rho \boldsymbol{U} \overbrace{(\boldsymbol{U} \cdot \hat{n})}^{U_{rn}} dA - \int_{in} \rho \boldsymbol{U} \overbrace{(\boldsymbol{U} \cdot \hat{n})}^{U_{rn}} dA \tag{6.16}$$

In situations where the velocity is provided and known (remember that density is constant) the integral can be replaced by

$$\boldsymbol{F} = \dot{m}\overline{\boldsymbol{U}_o} - \dot{m}\overline{\boldsymbol{U}_i} \tag{6.17}$$

The average velocity is related to the velocity profile by the following integral

$$\overline{U}^2 = \frac{1}{A} \int_A \left[U(r)\right]^2 dA \tag{6.18}$$

Equation (6.18) is applicable to any velocity profile and any geometrical shape.

Example 6.1:
*Calculate the average velocity for the given parabolic velocity profile for a circular pipe.*

SOLUTION

The velocity profile is

$$U\left(\frac{r}{R}\right) = U_{max}\left[1-\left(\frac{r}{R}\right)^2\right] \tag{6.I.a}$$

Substituting equation (6.I.a) into equation (6.18)

$$\overline{U}^2 = \frac{1}{2\,\pi\,R^2}\int_0^R \left[U(r)\right]^2\, 2\,\pi\,r\,dr \tag{6.I.b}$$

results in

$$\overline{U}^2 = \left(U_{max}\right)^2 \int_0^1 \left(1-\bar{r}^2\right)^2 \bar{r}d\bar{r} = \frac{1}{6}\left(U_{max}\right)^2 \tag{6.I.c}$$

Thus,

$$\overline{U} = \frac{U_{max}}{\sqrt{6}}$$

End Solution

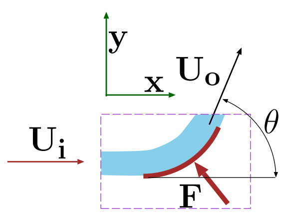



Fig a. *Schematics of area impinged by a jet for example 6.2.*

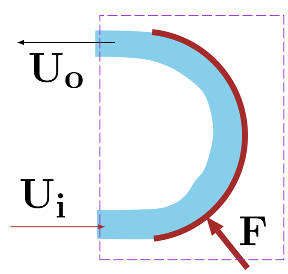



Fig b. *Schematics of maximum angle for impinged by a jet.*

*Fig. -6.2. Schematics of area impinged by a jet and angle effects.*

Example 6.2:
*A jet is impinging on a stationary surface by changing only the jet direction (see Figure 6.2). Neglect the friction, calculate the force and the angle which the support has to apply to keep the system in equilibrium. What is the angle for which maximum force will be created?*

Solution

Equation (6.11) can be reduced, because it is a steady state, to

$$\boldsymbol{F} = \int_{out} \rho \boldsymbol{U} \overbrace{(\boldsymbol{U} \cdot \hat{n})}^{U_{rn}} dA - \int_{in} \rho \boldsymbol{U} \overbrace{(\boldsymbol{U} \cdot \hat{n})}^{U_{rn}} dA = \dot{m} \boldsymbol{U_o} - \dot{m} \boldsymbol{U_i} \tag{6.II.a}$$

It can be noticed that even though the velocity change direction, the mass flow rate remains constant. Equation (6.II.a) can be explicitly written for the two coordinates. The equation for the $x$ coordinate is

$$F_x = \dot{m}\, (\cos\theta\, U_o - U_i)$$

or since $U_i = U_o$

$$F_x = \dot{m}\, U_i\, (\cos\theta - 1)$$

It can be observed that the maximum force, $F_x$ occurs when $\cos\theta = \pi$. It can be proven by setting $dF_x/d\theta = 0$ which yields $\theta = 0$ a minimum and the previous solution. Hence

$$F_x\big|_{max} = -2\, \dot{m}\, U_i$$

and the force in the $y$ direction is

$$F_y = \dot{m}\, U_i\, \sin\theta$$

the combined forces are

$$F_{total} = \sqrt{{F_x}^2 + {F_y}^2} = \dot{m}\, U_i \sqrt{\left(\cos\theta - 1\right)^2 + \sin^2\theta}$$

Which results in

$$F_{total} = \dot{m}\, U_i \sin\left(\theta/2\right)$$

with the force angle of

$$\tan\phi = \pi - \frac{F_y}{F_x} = \frac{\pi}{2} - \frac{\theta}{2}$$

For angle between $0 < \theta < \pi$ the maximum occur at $\theta = \pi$ and the minimum at $\theta \sim 0$. For small angle analysis is important in the calculations of flow around thin wings.

End Solution

Example 6.3:

*Liquid flows through a symmetrical nozzle as shown in the Figure 6.3 with a mass* *flow rate of 0.01 [$gk/sec$]. The entrance pressure is 3[$Bar$] and the entrance velocity is 5 [$m/sec$]. The exit velocity is uniform but unknown. The exit pressure is 1[$Bar$]. The entrance area is 0.0005[$m^2$] and the exit area is 0.0001[$cm^2$]. What is the exit velocity? What is the force acting the nozzle? Assume that the density is constant $\rho = 1000[kg/m^3]$ and the volume in the nozzle is 0.0015 [$m^3$].*

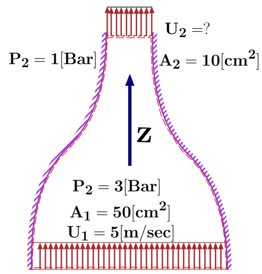



*Fig. -6.3. Nozzle schematic for the discussion on the forces and for example 6.3.*

Solution

The chosen control volume is shown in Figure 6.3. First, the velocity has to be found. This situation is a steady state for constant density. Then

$$A_1\,U_1 = A_2\,U_2$$

and after rearrangement, the exit velocity is

$$U_2 = \frac{A_1}{A_2}\,U_1 = \frac{0.0005}{0.0001} \times 5 = 25[m/sec]$$

Equation (6.12) is applicable but should be transformed into the $z$ direction which is

$$\sum F_z + \int_{c.v.} \boldsymbol{g}\cdot\hat{k}\,\rho\,dV + \int_{c.v.} \boldsymbol{P}\cos\theta_z\,dA + \int_{c.v.} \boldsymbol{\tau}_z\,dA = \overbrace{\frac{t}{dt}\int_{c.v.} \rho\boldsymbol{U}_z\,dV}^{=0} + \int_{c.v.} \rho\boldsymbol{U}_z\cdot\boldsymbol{U}_{rn}dA \tag{6.III.a}$$

The control volume does not cross any solid body (or surface) there is no external forces. Hence,

$$\overbrace{\sum F_z}^{=0} + \int_{c.v.} \boldsymbol{g}\cdot\hat{k}\,\rho\,dV + \overbrace{\int_{c.v.} \boldsymbol{P}\cos\theta_z\,dA}^{\text{liquid surface}} + \overbrace{\overbrace{\int_{c.v.} \boldsymbol{P}\cos\theta_z\,dA}^{\text{solid surface}} + \int_{c.v.} \boldsymbol{\tau}_z\,dA}^{\text{forces on the nozzle } F_{nozzle}} = \int_{c.v.} \rho\boldsymbol{U}_z\cdot\boldsymbol{U}_{rn}dA \tag{6.III.b}$$

All the forces that act on the nozzle are combined as

$$\sum F_{nozzle} + \int_{c.v.} \boldsymbol{g} \cdot \hat{k}\, \rho\, dV + \int_{c.v.} \boldsymbol{P} \cos\theta_z \, dA = \int_{c.v.} \rho \boldsymbol{U}_z \cdot \boldsymbol{U}_{rn} dA \tag{6.III.c}$$

The second term or the body force which acts through the center of the nozzle is

$$\boldsymbol{F}_b = -\int_{c.v.} \boldsymbol{g} \cdot \hat{n}\, \rho\, dV = -g\, \rho V_{nozzle}$$

Notice that in the results the gravity is not bold since only the magnitude is used. The part of the pressure which act on the nozzle in the $z$ direction is

$$-\int_{c.v.} P dA = \int_1 P dA - \int_2 P dA = \left. PA \right|_1 - \left. PA \right|_2$$

The last term in equation (6.III.c) is

$$\int_{c.v.} \rho \boldsymbol{U}_z \cdot \boldsymbol{U}_{rn} dA = \int_{A_2} U_2 \left(U_2\right) dA - \int_{A_1} U_1 \left(U_1\right) dA$$

which results in

$$\int_{c.v.} \rho \boldsymbol{U}_z \cdot \boldsymbol{U}_{rn} dA = \rho \left( {U_2}^2 A_2 - {U_1}^2 A_1 \right)$$

Combining all transform equation (6.III.c) into

$$F_z = -g\, \rho V_{nozzle} + \left. PA \right|_2 - \left. PA \right|_1 + \rho \left( {U_2}^2 A_2 - {U_1}^2 A_1 \right) \tag{6.III.d}$$

$$F_z = 9.8 \times 1000 \times$$

End Solution

## 6.2 Momentum Equation Application

**Momentum Equation Applied to Propellers**

The propeller is a mechanical devise that is used to increase the fluid momentum. Many times it is used for propulsion purposes of airplanes, ships and other devices (thrust) as shown in Figure 6.4. The propeller can be stationary like in cooling tours, fan etc. The other common used of propeller is mostly to move fluids as a pump.

The propeller analysis of unsteady is complicated due to the difficulty in understanding the velocity field. For a steady state the analysis is simpler and used here to provide an example of steady state. In the Figure 6.4 the fluid flows from the left to the right. Either it is assumed that some of the fluid enters into the container and fluid outside is not affected by the propeller. Or there is a line (or surface) in which the fluid outside changes only the flow direction. This surface is called slip surface. Of course it is only approximation but is provided a crude tool. Improvements can be made to this analysis. Here, this analysis is used for academic purposes.

As first approximation, the pressure around control volume is the same. Thus, pressure drops from the calculation. The one dimensional momentum equation is reduced

$$F = \rho \left( {U_2}^2 - {U_1}^2 \right) \tag{6.19}$$

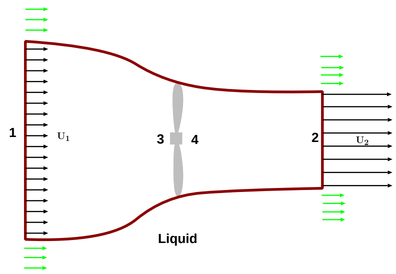



*Fig. -6.4. Propeller schematic to explain the change of momentum due to velocity.*

Combining the control volume between points 1 and 3 with (note that there are no external forces) with points 4 and 2 results in

$$\rho \left( {U_2}^2 - {U_1}^2 \right) = P_4 - P_3 \tag{6.20}$$

This analysis provide way to calculate the work needed to move this propeller. Note that in this analysis it was assumed that the flow is horizontal that $z_1 = z_2$ and/or the change is insignificant.

**Jet Propulsion**

Jet propulsion is a mechanism in which the air planes and other devices are propelled. Essentially, the air is sucked into engine and with addition heating (burning fuel) the velocity is increased. Further increase of the exit area with the increased of the burned gases further increase the thrust. The analysis of such device in complicated and there is a whole class dedicated for such topic in many universities. Here, a very limited discussion related to the steady state is offered.

The difference between the jets propulsion and propellers is based on the energy supplied. The propellers are moved by a mechanical work which is converted to thrust. In Jet propulsion, the thermal energy is converted to thrust. Hence, this direct conversion can be, and is, in many case more efficient. Furthermore, as it will be shown in the Chapter on compressible flow it allows to achieve velocity above speed of sound, a major obstacle in the past.

The inlet area and exit area are different for most jets and if the mass of the fuel is neglected then

$$F = \rho \left( A_2 \, {U_2}^2 - A_1 \, {U_1}^2 \right) \tag{6.21}$$

An academic example to demonstrate how a steady state calculations are done for a moving control volume. Notice that

Example 6.4:

*A sled toy shown in Figure 6.5 is pushed by liquid jet. Calculate the friction force on the* *toy when the toy is at steady state with velocity, $U_0$. Assume that the jet is horizontal and the reflecting jet is vertical. The velocity of the jet is uniform. Neglect the friction between the liquid (jet) and the toy and between the air and toy. Calculate the absolute velocity of the jet exit. Assume that the friction between the toy and surface (ground) is relative to the vertical force. The dynamics friction is $\mu_d$.*

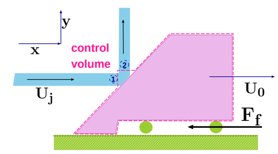



*Fig. -6.5. Toy Sled pushed by the liquid jet in a steady state for example 6.4.*

<u>Solution</u>

The chosen control volume is attached to the toy and thus steady state is obtained. The frame of reference is moving with the toy velocity, $\boldsymbol{U}_0$. The applicable mass conservation equation for steady state is

$$A_1 U_1 = A_2 U_2$$

The momentum equation in the $x$ direction is

$$\boldsymbol{F}_f + \int_{c.v.} \boldsymbol{g}\,\rho\, dV - \int_{c.v.} \boldsymbol{P}\, dA + \int_{c.v.} \boldsymbol{\tau}\, dA = \int_{c.v.} \rho \boldsymbol{U}\boldsymbol{U}_{rn} dV \tag{6.IV.a}$$

The relative velocity into the control volume is

$$\boldsymbol{U}_{1j} = (U_j - U_0)\ \hat{x}$$

The relative velocity out the control volume is

$$\boldsymbol{U}_{2j} = (U_j - U_0)\ \hat{y}$$

The absolute exit velocity is

$$\boldsymbol{U}_2 = U_0 \hat{x} + (U_j - U_0)\ \hat{y}$$

For small volume, the gravity can be neglected also because this term is small compared to other terms, thus

$$\int_{c.v.} \boldsymbol{g}\,\rho\, dV \sim 0$$

The same can be said for air friction as

$$\int_{c.v.} \boldsymbol{\tau}\, dA \sim 0$$

The pressure is uniform around the control volume and thus the integral is

$$\int_{c.v.} \boldsymbol{P}\, dA = 0$$

The control volume was chosen so that the pressure calculation is minimized.

The momentum flux is

$$\int_{S_{c.v.}} \rho\, U_x\, U_i rn\, dA = A\,\rho\, {U_{1j}}^2 \tag{6.IV.b}$$

The substituting (6.IV.b) into equation (6.IV.a) yields

$$F_f = A\,\rho\, {U_{1j}}^2 \tag{6.IV.c}$$

The friction can be obtained from the momentum equation in the $y$ direction

$$m_{toy}\, g + A\,\rho\, {U_{1j}}^2 = F_{earth}$$

According to the statement of question the friction force is

$$F_f = \mu_d \left(m_{toy}\, g + A\,\rho\, {U_{1j}}^2\right)$$

The momentum in the $x$ direction becomes

$$\mu_d \left(m_{toy}\, g + A\,\rho\, {U_{1j}}^2\right) = A\,\rho\, {U_{1j}}^2 = A\,\rho\, \left(U_j - U_0\right)^2$$

The toy velocity is then

$$U_0 = U_j - \sqrt{\frac{\mu_d\, m_{toy}\, g}{A\,\rho\, (1-\mu_d)}}$$

Increase of the friction reduce the velocity. Additionally larger toy mass decrease the velocity.

End Solution

### 6.2.1 Momentum for Unsteady State and Uniform Flow

The main problem in solving the unsteady state situation is that the control volume is accelerating. A possible way to solve the problem is by expressing the terms in an equation (6.10). This method is cumbersome in many cases. Alternative method of solution is done by attaching the frame of reference to the accelerating body. One such example of such idea is associated with the Rocket Mechanics which is present here.

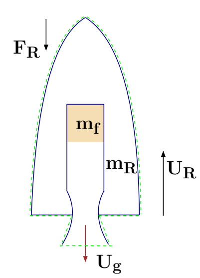



*Fig. -6.6. A rocket with a moving control volume.*

### 6.2.2 Momentum Application to Unsteady State

#### Rocket Mechanics

A rocket is a devise similar to jet propulsion. The difference is the fact that the oxidant is on board with the fuel. The two components are burned and the gases are ejected through a nozzle. This mechanism is useful for specific locations because it is independent of the medium though which it travels. In contrast to other mechanisms such as jet propulsion which obtain the oxygen from the medium which they travel the rockets carry the oxygen with it. The rocket is accelerating and thus the frame for reference is moving the with the rocket. The velocity of the rocket in the rocket frame of reference is zero. However, the derivative with respect to time, $d\boldsymbol{U}/dt \neq 0$ is not zero. The resistance of the medium is Denote as $F_R$. The momentum equation is

$$\overbrace{\int_{c.v.} \boldsymbol{\tau} dA}^{F_R} + \int_{c.v.} \boldsymbol{g}\,\rho\,dV + \overbrace{\int_{c.v.} \boldsymbol{P} dA}^{0} - \int \rho\,a_0\,dV = \frac{d}{dt}\int_{V_{c.v.}} \rho U_y dV + \int_{c.v.} \rho U_y\,U_{rn} dA \tag{6.22}$$

There are no external forces in this control volume thus, the first term $F_R$, vanishes. The pressure term vanish because the pressure essentially is the same and the difference can be neglected. The gravity term is an instantaneous mass times the gravity times the constant and the same can be said for the acceleration term. Yet, the acceleration is the derivative of the velocity and thus

$$\int \rho\,a_0\,dV = \frac{dU}{dt}\,(m_R + m_f) \tag{6.23}$$

The first term on the right hand side is the change of the momentum in the rocket volume. This change is due to the change in the volume of the oxidant and the fuel.

$$\frac{d}{dt}\int_{V_{c.v.}} \rho U_y dV = \frac{d}{dt}\left[(m_R + m_f)\;U\right] \tag{6.24}$$

Clearly, the change of the rocket mass can be considered minimal or even neglected. The oxidant and fuel flow outside. However, inside the rocket the change in the velocity is due to change in the reduction of the volume of the oxidant and fuel. This change is minimal and for this analysis, it can be neglected. The last term is

$$\int_{c.v.} \rho U_y\,U_{rn} dA = \dot{m}\,(U_g - U_R) \tag{6.25}$$

Combining all the above term results in

$$-F_R - (m_R + m_f)\;g + \frac{dU}{dt}\,(m_R + m_f) = \dot{m}\,(U_g - U_R) \tag{6.26}$$

Denoting $\mathcal{M}_T = m_R + m_f$ and thus $d\mathfrak{M}/dt = \dot{m}$ and $U_e = U_g - U_R$. As first approximation, for constant fuel consumption (and almost oxidant), gas flow out is constant as well. Thus, for constant constant gas consumption equation (6.26) transformed to

$$-F_R - \mathcal{M}_T\, g + \frac{dU}{dt}\, \mathcal{M}_T = \dot{\mathcal{M}}_T\, U_e \tag{6.27}$$

Separating the variables equation (6.27) yields

$$dU = \left( \frac{-\dot{\mathcal{M}}_T\, U_e}{\mathcal{M}_T} - \frac{F_R}{\mathcal{M}_T} - g \right) dt \tag{6.28}$$

Before integrating equation (6.28), it can be noticed that the friction resistance $F_R$, is a function of the several parameters such the duration, the speed (the Reynolds number), material that surface made and the medium it flow in altitude. For simplicity here the part close to Earth (to the atmosphere) is assumed to be small compared to the distance in space. Thus it is assume that $F_R = 0$. Integrating equation (6.28) with limits of $U(t=0) = 0$ provides

$$\int_0^U dU = -\dot{\mathcal{M}}_T\, U_e \int_0^t \frac{dt}{\mathcal{M}_T} - \int_0^t g\, dt \tag{6.29}$$

the results of the integration is (notice $\mathcal{M} = \mathcal{M}_0 - t\, \dot{\mathcal{M}}$)

$$U = U_e \ln \left( \frac{\mathcal{M}_0}{\mathcal{M}_0 - t\, \dot{\mathcal{M}}} \right) - g\, t \tag{6.30}$$

The following is an elaborated example which deals with an unsteady two dimensional problem. This problem demonstrates the used of control volume to find method of approximation for not given velocity profiles[1]

Example 6.5:

[1]A variation of this problem has appeared in many books in the literature. However, in the past it was not noticed that a slight change in configuration leads to a constant $x$ velocity. This problem was aroused in manufacturing industry. This author was called for consultation and to solve a related problem. For which he noticed this "constant velocity."

*A tank with wheels is filled with liquid is depicted in Figure 6.7. The tank upper part is opened to the atmosphere. At initial time the valve on the tank is opened and the liquid flows out with an uniform velocity profile. The tank mass with the wheels (the solid parts) is known, $m_t$. Calculate the tank velocity for two cases. One the wheels have a constant resistance with the ground and two the resistance linear function of the weight. Assume that the exit velocity is a linear function of the height.*

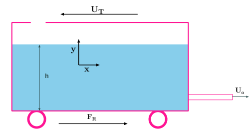



*Fig. -6.7. Schematic of a tank seating on wheel for unsteady state discussion*

SOLUTION

This problem is similar to the rocket mechanics with a twist, the source of the propulsion is the potential energy. Furthermore, the fluid has two velocity components verse one component in the rocket mechanics. The control volume is shown in Figure 6.7. The frame of reference is moving with the tank. This situation is unsteady state thus equation (6.12) for two dimensions is used. The mass conservation equation is

$$\frac{d}{dt}\int_{V_{c.v.}} \rho\, dV + \int_{S_{c.v.}} \rho dA = 0 \tag{6.V.a}$$

Equation (6.V.a) can be transferred to

$$\frac{dm_{c.v.}}{dt} = -\rho\, U_0\, A_0 = -m_0 \tag{6.V.b}$$

Where $m_0$ is mass flow rate out. Equation (6.V.b) can be further reduced due to constant density to

$$\frac{d\,(A\,h)}{dt} + U_0\, A_0 = 0 \tag{6.V.c}$$

It can be noticed that the area of the tank is almost constant ($A = constant$) thus

$$A\frac{dh}{dt} + U_0\, A_0 = 0 \Longrightarrow \frac{dh}{dt} = -\frac{U_0\, A_0}{A} \tag{6.31}$$

The relationship between the height and the flow now can be used.

$$U_0 = \mathcal{B}\, h \tag{6.V.d}$$

Where $\mathcal{B}$ is the coefficient that has the right units to mach equation (6.V.d) that represent the resistance in the system and substitute the energy equation. Substituting equation (6.V.d) into equation (6.V.c) results in

$$\frac{dh}{dt} + \frac{\mathcal{B}\, h\, A_0}{A} = 0 \tag{6.V.e}$$

Equation (6.V.e) is a first order differential equation which can be solved with the initial condition $h(t=0)=h_0$. The solution (see for details in the Appendix A.2.1 ) is

$$h(t) = h_0\, \mathsf{e}^{-\dfrac{t\,A_0\,\mathcal{B}}{A}} \tag{6.V.f}$$

To find the average velocity in the $x$ direction a new control volume is used. The boundary of this control volume are the tank boundary on the left with the straight surface as depicted in Figure 6.8. The last boundary is variable surface in a distance $x$ from the tank left part. The tank depth, is not relevant. The mass conservation for this control volume is

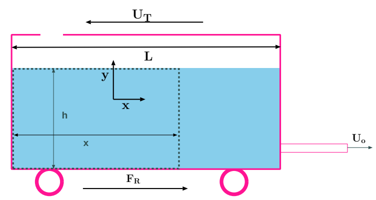



*Fig. -6.8. A new control volume to find the velocity in discharge tank for example 6.5.*

$$\cancel{w}\, x\, \frac{dh}{dt} = -\cancel{w}\, h\, \overline{U_x} \tag{6.V.g}$$

Where here $w$ is the depth or width of the tank. Substituting (6.V.f) into (6.V.g) results

$$\overline{U_x}(x) = \frac{x\,A_0\,\cancel{h_0}\mathcal{B}}{A\,\cancel{h}}\cancel{\mathsf{e}^{-\dfrac{t\,A_0\,\mathcal{B}}{A}}} = \frac{x\,A_0\mathcal{B}}{A} \tag{6.V.h}$$

The average $x$ component of the velocity is a linear function of $x$. Perhaps surprising, it also can be noticed that $\overline{U_x}(x)$ is a not function of the time. Using this function, the average velocity in the tank is

$$\overline{U_x} = \frac{1}{L}\int_0^L \frac{x\,A_0\,\mathcal{B}}{A} = \frac{L\,A_0\,\mathcal{B}}{2\,A} \tag{6.V.i}$$

It can be noticed that $\overline{U_x}$ is not function of height, $h$. In fact, it can be shown that average velocity is a function of cross section (what direction?).

Using a similar control volume[2], the average velocity in the $y$ direction is

$$\overline{U_y} = \frac{dh}{dt} = -\frac{h_0\,A_0\,\mathcal{B}}{A}\mathsf{e}^{-\dfrac{t\,A_0\,\mathcal{B}}{A}} \tag{6.V.j}$$

It can be noticed that the velocity in the $y$ is a function of time as oppose to the $x$ direction.

The applicable momentum equation (in the tank frame of reference) is (6.11) which is reduced to

$$-\boldsymbol{F}_R - (m_t+m_f)\,\boldsymbol{g} - \overbrace{\boldsymbol{a}\,(m_t+m_f)}^{acceleration} = \frac{d}{dt}\left[(m_t+m_f)\,\boldsymbol{U}_r\right] + U_0\,m_o \tag{6.V.k}$$

[2]The boundaries are the upper (free surface) and tank side with a $y$ distance from the free surface. $\int U_{bn}dA = \int U_{rn}dA \Longrightarrow U_{bn} = U_{rn}$.

Where $\boldsymbol{U}_r$ is the relative fluid velocity to the tank (if there was no tank movement). $m_f$ and $m_t$ are the mass of the fluid and the mass of tank respectively. The acceleration of the tank is $\boldsymbol{a} = -\hat{\imath} a_0$ or $\hat{\imath} \cdot \boldsymbol{a} = -a$. And the additional force for accelerated system is

$$-\hat{\imath} \cdot \int_{V_{c.v.}} \boldsymbol{a} \rho dV = m_{c.v.}\, a$$

The mass in the control volume include the mass of the liquid with mass of the solid part (including the wheels).

$$m_{c.v.} = m_f + m_T$$

because the density of the air is very small the change of the air mass is very small as well ($\rho_a << \rho$).

The pressure around the control volume is uniform thus

$$\int_{S_{c.v.}} P \cos\theta_x dA \sim 0$$

and the resistance due to air is negligible, hence

$$\int_{S_{c.v.}} \boldsymbol{\tau} dA \sim 0$$

The momentum flow rate out of the tank is

$$\int_{S_{c.v.}} \rho\, U_x\, U_{rn} dA = \rho\, {U_o}^2 A_o = m_o\, U_o \tag{6.32}$$

In the $x$ coordinate the momentum equation is

$$-F_x + (m_t + m_f)\, a = \frac{d}{dt}\left[(m_t + m_f)\; U_x\right] + U_0\, \dot{m}_f \tag{6.V.l}$$

Where $F_x$ is the $x$ component of the reaction which is opposite to the movement direction. The momentum equation in the $y$ coordinate it is

$$F_y - (m_t + m_f)\, g = \frac{d}{dt}\left[\,(m_t + m_f)\; U_y\right] \tag{6.V.m}$$

There is no mass flow in the $y$ direction and $U_y$ is component of the velocity in the $y$ direction.

The tank movement cause movement of the air which cause momentum change. This momentum is function of the tank volume times the air density times tank velocity ($h_0 \times A \times \rho_a \times U$). This effect is known as the add mass/momentum and will be discussed in the Dimensional Analysis and Ideal Flow Chapters. Here this effect is neglected.

The main problem of integral analysis approach is that it does not provide a way to analysis the time derivative since the velocity profile is not given inside the control volume. This limitation can be partially overcome by assuming some kind of average. It can be noticed that the velocity in the tank has two components. The first component is downward ($y$) direction and the second in the exit direction ($x$). The velocity in the $y$ direction does not contribute to the momentum in the $x$ direction. The average velocity in the tank (because constant density and more about it later section) is

$$\overline{U_x} = \frac{1}{V_t}\int_{V_f} U_x dV$$

Because the integral is replaced by the average it is transferred to

$$\int_{V_f} \rho\, U_x dV \sim m_{c.v.}\, \overline{U_x}$$

Thus, if the difference between the actual and averaged momentum is neglected then

$$\frac{d}{dt}\int_{V_f} \rho\, U_x\, dV \sim \frac{d}{dt}\left(m_{c.v.}\,\overline{U_x}\right) = \frac{d\, m_{c.v.}}{dt}\overline{U_x} + \overbrace{\frac{d\,\overline{U_x}}{dt}}^{\sim 0} m_{c.v.} \tag{6.V.n}$$

Noticing that the derivative with time of control volume mass is the flow out in equation (6.V.n) becomes

$$\frac{d\, m_{c.v.}}{dt}\overline{U_x} + \frac{d\,\overline{U_x}}{dt} m_{c.v.} = -\overbrace{\dot{m}_0}^{\text{mass rate out}}\ \overline{U_x} = -m_0\,\frac{L\, A_0\, \mathcal{B}}{2\, A} \tag{6.V.o}$$

Combining all the terms results in

$$-F_x + a\ (m_f + m_t) = -m_0\,\frac{L\, A_0\, \mathcal{B}}{2\, A} - U_0\, m_0 \tag{6.V.p}$$

Rearranging and noticing that $a = dU_T/dt$ transformed equation (6.V.p) into

$$a = \frac{F_x}{m_f + m_t} - m_0\left(\frac{L\, A_0\, \mathcal{B} + 2\, A\, U_0\,(m_f + m_t)}{2\, A\,(m_f + m_t)}\right) \tag{6.V.q}$$

If the $F_x \geq m_0\left(\frac{L\, A_0\, \mathcal{B}}{2\, A} + U_0\right)$ the toy will not move. However, if it is the opposite the toy start to move. From equation (6.V.d) the mass flow out is

$$m_0(t) = \overbrace{\mathcal{B}\ \overbrace{h_0\, \mathrm{e}^{-\dfrac{t\, A_0\, \mathcal{B}}{A}}}^{h}}^{U_0} A_0\, \rho \tag{6.V.r}$$

The mass in the control volume is

$$m_f = \rho \overbrace{A\,h_0\,\mathsf{e}^{-\dfrac{t\,A_0\,\mathcal{B}}{A}}}^{V} \tag{6.V.s}$$

The initial condition is that $U_T(t=0)=0$. Substituting equations (6.V.r) and (6.V.s) into equation (6.V.q) transforms it to a differential equation which is integrated if $R_x$ is constant.

For the second case where $R_x$ is a function of the $R_y$ as

$$R_x = \mu\,R_y \tag{6.33}$$

The $y$ component of the average velocity is function of the time. The change in the accumulative momentum is

$$\frac{d}{dt}\left[\,(m_f)\ \overline{U_y}\right] = \frac{dm_f}{dt}\overline{U_y} + \frac{d\overline{U_y}}{dt}m_f \tag{6.V.t}$$

The reason that $m_f$ is used because the solid parts do not have velocity in the $y$ direction. Rearranging the momentum equation in the $y$ direction transformed

$$F_y = \left(m_t + \overbrace{\rho\,A\,h_0\mathsf{e}^{-\dfrac{t\,A_0\,\mathcal{B}}{A}}}^{m_f}\right) g + 2\left(\frac{\rho\,h_0 {A_0}^2\,\mathcal{B}^2}{A}\right)^2 \mathsf{e}^{-\dfrac{t\,A_0\,\mathcal{B}}{A}} \tag{6.V.u}$$

The actual results of the integrations are not provided since the main purpose of this exercise to to learn how to use the integral analysis.

End Solution

**Averaged Velocity! Estimates**

In example 6.1 relationship between momentum of maximum velocity to average velocity was presented. Here, relationship between momentum for the average velocity to the actual velocity is presented. There are situations where actual velocity profile is not known but is function can be approximated. For example, the velocity profile can be estimated using the ideal fluid theory but the actual values are not known. For example, the flow profile in example 6.5 can be estimated even by hand sketching.

For these cases a correction factor can be used. This correction factor can be calculated by finding the relation between the two cases. The momentum for average velocity is

$$M_a = m_{c.v}\overline{U} = \rho\,V\int_{c.v} U dV \tag{6.34}$$

The actual momentum for control volume is

$$M_c = \int_{c.v.} \rho\,U_x dV \tag{6.35}$$

These two have to equal thus,

$$\mathcal{C}\,\rho\,V \int_{c.v} U dV = \int_{c.v.} \rho\,U_x dV \tag{6.36}$$

If the density is constant then the coefficient is one ($\mathcal{C} \equiv 1$). However, if the density is not constant, the coefficient is not equal to one.

## 6.3 Conservation Moment Of Momentum

The angular momentum can be derived in the same manner as the momentum equation for control volume. The force

$$F = \frac{D}{Dt} \int_{V_{sys}} \rho \boldsymbol{U} dV \tag{6.37}$$

The angular momentum then will be obtained by calculating the change of every element in the system as

$$\mathfrak{M} = \boldsymbol{r} \times \boldsymbol{F} = \frac{D}{Dt} \int_{V_{sys}} \rho\, \boldsymbol{r} \times \boldsymbol{U}\, dV \tag{6.38}$$

Now the left hand side has to be transformed into the control volume as

$$\mathfrak{M} = \frac{d}{dt} \int_{V_{c.v.}} \rho\, \left(\boldsymbol{r} \times \boldsymbol{U}\right)\, dV + \int_{S_{c.v}} \rho\, \left(\boldsymbol{r} \times \boldsymbol{U}\right) \boldsymbol{U}_{rn}\, dA \tag{6.39}$$

The angular momentum equation, applying equation (6.39) to uniform and steady state flow with neglected pressure gradient is reduced to

$$\mathfrak{M} = \dot{m}\left(r_2 \times U_2 + r_2 \times U_1\right) \tag{6.40}$$

**Introduction to Turbo Machinery**

The analysis of many turbomachinary such as centrifugal pump is fundamentally based on the angular momentum. To demonstrate this idea, the following discussion is provided. A pump impeller is shown in Figure 6.9 commonly used in industry. The impeller increases the velocity of the fluid by increasing the radius of the particles. The inside particle is obtained larger velocity and due to centrifugal forces is moving to outer radius for which additionally increase of velocity occur. The pressure on the outer side is uniform thus does not create a moment. The flow is

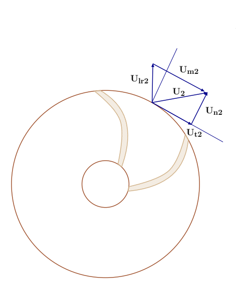



*Fig. -6.9. The impeller of the centrifugal pump and the velocities diagram at the exit.*

assumed to enter the impeller radially with average velocity $U_1$. Here it is assumed that fluid is incompressible ($\rho = constant$). The height of the impeller is $h$. The exit liquid velocity, $U_2$ has two components, one the tangential velocity, $U_{t2}$ and radial component, $U_{n2}$. The relative exit velocity is $U_{lr2}$ and the velocity of the impeller edge is $U_{m2}$. Notice that tangential liquid velocity, $U_{t2}$ is not equal to the impeller outer edge velocity $U_{m2}$. It is assumed that required torque is function $U_2$, $r$, and $h$.

$$\mathfrak{M} = \dot{m}\, r_2\, U_{t2} \tag{6.41}$$

Multiplying equation (6.41) results in

$$\mathfrak{M}\,\omega = \dot{m}\, \overbrace{r_2\,\omega}^{U_{m2}}\, U_{t2} \tag{6.42}$$

The shaft work is given by the left side and hence,

$$\dot{W} = \dot{m}\, U_{m2}\, U_{t2} \tag{6.43}$$

The difference between $U_{m2}$ to $U_{t2}$ is related to the efficiency of the pump which will be discussed in the chapter on the turbomachinary.

Example 6.6:
*A centrifugal pump is pumping 600* $2[m^3/hour]$. *The thickness of the impeller,* $h$ *is* $2[cm]$ *and the exit diameter is* $0.40[m]$. *The angular velocity is 1200* $r.p.m.$ *Assume that angle velocity is leaving the impeller is* $125^\circ$. *Estimate what is the minimum energy required by the pump.*

## 6.4 More Examples on Momentum Conservation

Example 6.7:
*A design of a rocket is based on the idea that density increase of the leaving jet increases the acceleration of the rocket see Figure*

*6.10. Assume that this idea has a good engineering logic. Liquid fills the lower part of the rocket tank. The upper part of the rocket tank is filled with compressed gas. Select the control volume in such a way that provides the ability to find the rocket acceleration. What is the instantaneous velocity of the rocket at time zero? Develop the expression for the pressure (assuming no friction with the walls). Develop expression for rocket velocity. Assume that the gas is obeying the perfect gas model. What are the parameters that effect the problem.*

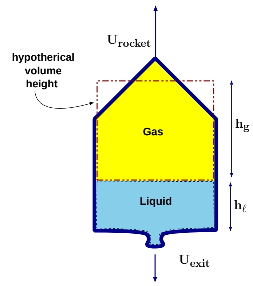



*Fig. -6.10. Nozzle schematics water rocket for the discussion on the forces for example 6.7*

SOLUTION

**Under construction for time being only hints**[3]

In the solution of this problem several assumptions must be made so that the integral system can be employed.

- The surface remained straight at the times and no liquid residue remains behind.
- The gas obeys the ideal gas law.
- The process is isothermal (can be isentropic process).
- No gas leaves the rocket.
- The mixing between the liquid and gas is negligible.
- The gas mass is negligible in comparison to the liquid mass and/or the rocket.
- No resistance to the rocket (can be added).
- The cross section of the liquid is constant.

In this problem the energy source is the pressure of the gas which propels the rocket. Once the gas pressure reduced to be equal or below the outside pressure the rocket have no power for propulsion. Additionally, the initial take off is requires a larger pressure.

The mass conservation is similar to the rocket hence it is

$$\frac{dm}{dt} = -U_e\,A_e \tag{6.VII.a}$$

[3]This problem appeared in the previous version (0.2.3) without a solution. Several people ask to provide a solution or some hints for the solution. The following is not the solution but rather the approach how to treat this problem.

The mass conservation on the gas zone is a byproduct of the mass conservation of the liquid. Furthermore, it can be observed that the gas pressure is a direct function of the mass flow out.

The gas pressure at the initial point is

$$P_0 = \rho_0 \, R \, T \tag{6.VII.b}$$

Per the assumption the gas mass remain constant and is denoted as $m_g$. Using the above definition, equation (6.VII.b) becomes

$$P_0 = \frac{m_g \, R \, T}{V_{0g}} \tag{6.VII.c}$$

The relationship between the gas volume

$$V_g = \overline{h_g} \, A \tag{6.VII.d}$$

The gas geometry is replaced by a virtual constant cross section which cross section of the liquid (probably the same as the base of the gas phase). The change of the gas volume is

$$\frac{dV_g}{dt} = A \, \frac{dh_g}{dt} = -A \, \frac{dh_\ell}{dt} \tag{6.VII.e}$$

The last identify in the above equation is based on the idea what ever height concede by the liquid is taken by the gas. The minus sign is to account for change of "direction" of the liquid height. The total change of the gas volume can be obtained by integration as

$$V_g = A \, \left(h_{g0} - \Delta h_\ell\right) \tag{6.VII.f}$$

It must be point out that integral is not function of time since the height as function of time is known at this stage.

The initial pressure now can be expressed as

$$P_0 = \frac{m_g \, R \, T}{h_{g0} \, A} \tag{6.VII.g}$$

The pressure at any time is

$$P = \frac{m_g \, R \, T}{h_g \, A} \tag{6.VII.h}$$

Thus the pressure ratio is

$$\frac{P}{P_0} = \frac{h_{g0}}{h_g} = \frac{h_{g0}}{h_{g0} - \Delta h_\ell} = h_{g0} \frac{1}{1 - \dfrac{\Delta h_\ell}{h_{g0}}} \tag{6.VII.i}$$

Equation (6.VII.a) can be written as

$$m_\ell(t) = m_{\ell 0} - \int_0^t U_e \, A_e dt \tag{6.VII.j}$$

From equation (6.VII.a) it also can be written that

$$\frac{dh_\ell}{dt} = \frac{U_e\,A_e}{\rho_e\,A} \tag{6.VII.k}$$

According to the assumption the flow out is linear function of the pressure inside thus,

$$U_e = f(P) + g\,h_\ell\,rho \backsimeq f(P) = \zeta\,P \tag{6.VII.l}$$

Where $\zeta$ here is a constant which the right units.

The liquid momentum balance is

$$-g\,(m_R + m_\ell) - a\,(m_R + m_\ell) = \overbrace{\frac{d}{dt}\,(m_R + m_\ell)\,U}^{=0} + bc + (U_R + U_\ell)\,m_\ell \tag{6.VII.m}$$

Where $bc$ is the change of the liquid mass due the boundary movement.

End Solution

Example 6.8:

*A rocket is filled with only compressed gas. At a specific moment the valve is opened and the rocket is allowed to fly. What is the minimum pressure which make the rocket fly. What are the parameters that effect the rocket velocity. Develop an expression for the rocket velocity.*

Example 6.9:

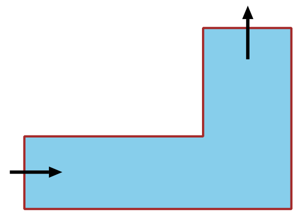
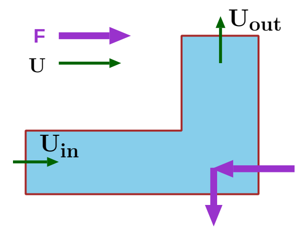
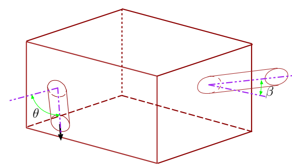
*In Example 6.5 it was mentioned that there are only two velocity components. What was the assumption that the third velocity component was neglected.*

### 6.4.1 Qualitative Questions

Example 6.10:

*For each following figures discuss and state force direction and the momentum that act on the control volume due to .*

Example 6.11:

| **Situations** | **Explanations** |
|---|---|
|  Flow in and out of Angle |   |
|  Flow in and out at angle from a tank | |

*A similar tank as shown in Figure 6.11 is built with a exit located in uneven distance from the the right and the left and is filled with liquid. The exit is located on the left hand side at the front. What are the direction of the forces that keep the control volume in the same location? Hints, consider the unsteady effects. Look at the directions which the unsteady state momentum in the tank change its value.*

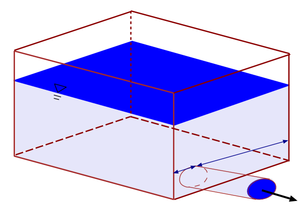


*Fig. -6.11. Flow out of un symmetrical tank for example 6.11*

Example 6.12:
*A large tank has opening with area, $A$. In front and against the opening there a block with mass of 50$[kg]$. The friction factor between the block and surface is 0.5. Assume that resistence between the air and the water jet is negligible. Calculated the minimum height of the liquid in the tank in order to start to have the block moving?*

<u>Solution</u>

The solution of this kind problem first requires to know at what accuracy this solution is needed. For great accuracy, the effect minor loss or the loss in the tank opening have taken into account. First assuming that a minimum accuracy therefore the infomration was given on the tank that it large. First, the velocity to move the block can be obtained from the analysis of the block free body diagram (the impeging jet diagram).

The control volume is attached to the block. It is assumed that the two streams in the vertical cancle each other. The jet stream has only one componet in the horizontal component. Hence,

$$F = \rho\, A\, {U_{exit}}^2 \tag{6.XII.a}$$

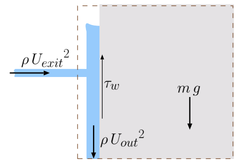



*Fig. -6.12. Jet impinging jet surface perpendicular and with the surface.*

The miminum force the push the plock is

$$\rho\, A\, {U_{exit}}^2 = m\, g\, \mu \Longrightarrow U_{exit} = \sqrt{\frac{m\, g\, \mu}{\rho\, A}} \tag{6.XII.b}$$

And the velocity as a function of the height is $U = \sqrt{\rho\, g\, h}$ and thus

$$h = \frac{m\, \mu}{\rho^2\, A} \tag{6.XII.c}$$

It is interesting to point out that the gravity is relavent. That is the gravity has no effect on the velocity (height) required to move the block. However, if the gravity was in the opposite direction, no matter what the height will be the block will not move (neglecting other minor effects). So, the gravity has effect and the effect is the direction, that is the same height will be required on the moon as the earth.

For very tall blocks, the forces that acts on the block in the vertical direction is can be obtained from the analysis of the control volume shown in Figure 6.12. The jet impenged on the surface results in out flow stream going to all the directions in the block surface. Yet, the gravity acts on all these "streams" and eventually the liquid flows downwards. In fact because the gravity the jet impeging in downwards slend direction. At the exreme case, all liquid flows downwords. The balance on the stream downwords (for steady state) is

$$\rho\, \overline{U_{out}}^2 \cong \rho\, V_{liquid}\, g + m\, g \tag{6.XII.d}$$

Where $V_{liquid}$ is the liquid volume in the control volume (attached to the block). The pressure is canceled because the flow is exposed to air. In cases were $\rho\, V_{liquid}\, g > \rho\, \overline{U_{out}}^2$ the required height is larger. In the oposite cases the height is smaller.

End Solution

# CHAPTER 7

# Energy Conservation

## 7.1 The First Law of Thermodynamics

This chapter focuses on the energy conservation which is the first law of thermodynamics[1]. The fluid, as all phases and materials, obeys this law which creates strange and wonderful phenomena such as a shock and choked flow. Moreover, this law allows to solve problems, which were assumed in the previous chapters. For example, the relationship between height and flow rate was assumed previously, here it will be derived. Additionally a discussion on various energy approximation is presented.

It was shown in Chapter 2 that the energy rate equation (2.10) for a system is

$$\dot{Q} - \dot{W} = \frac{D\,E_U}{Dt} + \frac{D\left(m\,U^2\right)}{Dt} + \frac{D\,(m\,g\,z)}{Dt} \tag{7.1}$$

This equation can be rearranged to be

$$\dot{Q} - \dot{W} = \frac{D}{Dt}\left(E_U + m\,\frac{U^2}{2} + m\,g\,z\right) \tag{7.2}$$

Equation (7.2) is similar to equation (6.3) in which the right hand side has to be interpreted and the left hand side interpolated using the Reynold's Transport Theorem (RTT)[2]. The right hand side is very complicated and only some of the effects will be discussed (It is only an introductory material).

[1]Thermodynamics is the favorite topic of this author since it was his major in high school. Clearly this topic is very important and will be extensively discussed here. However, during time of the constructing this book only a simple skeleton by Potto standards will be build.

[2]Some view the right hand side as external effects while the left side of the equation represents the internal effects. This simplistic representation is correct only under extreme conditions. For example, the above view is wrong when the heat convection, which is external force, is included on the right hand side.

The energy transfer is carried (mostly[3]) by heat transfer to the system or the control volume. There are three modes of heat transfer, conduction, convection[4] and radiation. In most problems, the radiation is minimal. Hence, the discussion here will be restricted to convection and conduction. Issues related to radiation are very complicated and considered advance material and hence will be left out. The issues of convection are mostly covered by the terms on the left hand side. The main heat transfer mode on the left hand side is conduction. Conduction for most simple cases is governed by Fourier's Law which is

$$d\dot{q} = k_T \frac{dT}{dn} dA \tag{7.3}$$

Where $d\dot{q}$ is heat transfer to an infinitesimal small area per time and $k_T$ is the heat conduction coefficient. The heat derivative is normalized into area direction. The total heat transfer to the control volume is

$$\dot{Q} = \int_{A_{cv}} k \frac{dT}{dn} dA \tag{7.4}$$

The work done on the system is more complicated to express than the heat transfer. There are two kinds of works that the system does on the surroundings. The first kind work is by the friction or the shear stress and the second by normal force. As in the previous chapter, the surface forces are divided into two categories: one perpendicular to the surface and one with the surface direction. The work done by system on the surroundings (see Figure 7.1) is

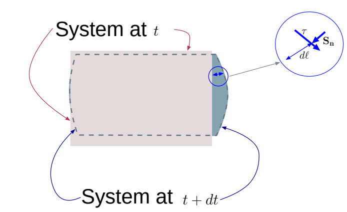



*Fig. -7.1. The work on the control volume is done by two different mechanisms, $\mathbf{S_n}$ and $\tau$.*

$$dw = \overbrace{-\boldsymbol{S}\, d\boldsymbol{A}}^{d\boldsymbol{F}} \cdot d\ell = -\left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \cdot \overbrace{d\boldsymbol{\ell} dA}^{dV} \tag{7.5}$$

The change of the work for an infinitesimal time (excluding the shaft work) is

$$\frac{dw}{dt} = -\left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \cdot \overbrace{\frac{d\boldsymbol{\ell}}{dt}}^{U} dA = -\left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \cdot \boldsymbol{U}\, dA \tag{7.6}$$

The total work for the system including the shaft work is

$$\dot{W} = -\int_{Ac.v.} \left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \boldsymbol{U}\, dA - W_{shaft} \tag{7.7}$$

[3]There are other methods such as magnetic fields (like microwave) which are not part of this book.
[4]When dealing with convection, actual mass transfer must occur and thus no convection is possible to a system by the definition of system.

The energy equation (7.2) for system is

$$\begin{aligned}\int_{A_{sys}} k_T \frac{dT}{dn} dA + & \int_{A_{sys}} (\boldsymbol{S_n} + \boldsymbol{\tau})\, dV \\ & + \dot{W}_{shaft} = \frac{D}{Dt} \int_{V_{sys}} \rho \left( E_U + m \frac{U^2}{2} + g\, z \right) dV\end{aligned} \tag{7.8}$$

Equation (7.8) does not apply any restrictions on the system. The system can contain solid parts as well several different kinds of fluids. Now Reynolds Transport Theorem can be used to transformed the left hand side of equation (7.8) and thus yields

Energy Equation

$$\begin{aligned}\int_{A_{cv}} k_T \frac{dT}{dn} dA + & \int_{A_{cv}} (\boldsymbol{S_n} + \boldsymbol{\tau})\, dA + \dot{W}_{shaft} = \\ & \frac{d}{dt} \int_{V_{cv}} \rho \left( E_u + m \frac{U^2}{2} + g\, z \right) dV \\ & + \int_{A_{cv}} \left( E_u + m \frac{U^2}{2} + g\, z \right) \rho\, U_{rn} dA\end{aligned} \tag{7.9}$$

From now on the notation of the control volume and system will be dropped since all equations deals with the control volume. In the last term in equation (7.9) the velocity appears twice. Note that $U$ is the velocity in the frame of reference while $U_{rn}$ is the velocity relative to the boundary. As it was discussed in the previous chapter the normal stress component is replaced by the pressure (see equation (6.8) for more details). The work rate (excluding the shaft work) is

$$\dot{W} \cong \overbrace{\int_S P \hat{n} \cdot \boldsymbol{U} dA}^{\text{flow work}} - \int_S \boldsymbol{\tau} \cdot \boldsymbol{U}\, \hat{n}\, dA \tag{7.10}$$

The first term on the right hand side is referred to in the literature as the flow work and is

$$\int_S P \hat{n} \cdot \boldsymbol{U} dA = \int_S P \overbrace{(U - U_b)\, \hat{n}}^{U_{rn}} dA + \int_S P\, U_{bn} dA \tag{7.11}$$

Equation (7.11) can be further manipulated to become

$$\int_S P \hat{n} \cdot \boldsymbol{U} dA = \overbrace{\int_S \frac{P}{\rho}\, \rho\, U_{rn}\, dA}^{\text{work due to the flow}} + \overbrace{\int_S P U_{bn} dA}^{\text{work due to boundaries movement}} \tag{7.12}$$

The second term is referred to as the shear work and is defined as

$$\dot{W}_{shear} = -\int_S \boldsymbol{\tau} \cdot \boldsymbol{U} dA \tag{7.13}$$

Substituting all these terms into the governing equation yields

$$\begin{aligned}\dot{Q} - \dot{W}_{shear} - \quad \dot{W}_{shaft} &= \frac{d}{dt}\int_V \left(E_u + \frac{U^2}{2} + g\,z\right) dV + \\ &\int_S \left(E_u + \frac{P}{\rho} + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\,dA + \int_S P U_{rn} dA\end{aligned} \tag{7.14}$$

The new term $P/\rho$ combined with the internal energy, $E_u$ is referred to as the enthalpy, $h$, which was discussed on page 48. With these definitions equation (7.14) transformed

Simplified Energy Equation

$$\boxed{\begin{aligned}\dot{Q} - \dot{W}_{shear} + \quad \dot{W}_{shaft} &= \frac{d}{dt}\int_V \left(E_u + \frac{U^2}{2} + g\,z\right) \rho\,dV + \\ &\int_S \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\,dA + \int_S P U_{bn} dA\end{aligned}} \tag{7.15}$$

Equation (7.15) describes the energy conservation for the control volume in stationary coordinates. Also note that the shear work inside the the control volume considered as shaft work.

The example of flow from a tank or container is presented to demonstrate how to treat some of terms in equation (7.15).

### Flow Out From A Container

In the previous chapters of this book, the flow rate out of a tank or container was assumed to be a linear function of the height. The flow out is related to the height but in a more complicate function and is the focus of this discussion. The energy equation with mass conservation will be utilized for this analysis. In this analysis several assumptions are made which includes the following: constant density, the gas density is very small compared to liquid density, and exit area is relatively small, so the velocity can be assumed uniform (not a function of the opening height)[5], surface tension effects are negligible and

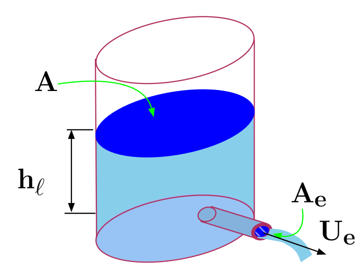



*Fig. -7.2. Discharge from a Large Container with a small diameter.*

[5]Later a discussion about the height opening effects will be discussed.

the liquid surface is straight[6]. Additionally, the temperature is assumed to constant. The control volume is chosen so that all the liquid is included up to exit of the pipe. The conservation of the mass is

$$\frac{d}{dt}\int_V \not\rho\, dV + \int_A \not\rho\, U_{rn}\, dA = 0 \tag{7.16}$$

which also can be written (because $\frac{d\rho}{dt} = 0$) as

$$\int_A U_{bn}\, dA + \int_A U_{rn} dA = 0 \tag{7.17}$$

Equation (7.17) provides the relationship between boundary velocity to the exit velocity as

$$A\, U_b = A_e\, U_e \tag{7.18}$$

Note that the boundary velocity is not the averaged velocity but the actual velocity. The averaged velocity in $z$ direction is same as the boundary velocity

$$U_b = U_z = \frac{dh}{dt} = \frac{A_e}{A}\, U_e \tag{7.19}$$

The $x$ component of the averaged velocity is a function of the geometry and was calculated in Example 5.12 to be larger than

$$\overline{U_x} \precapprox \frac{2\,r}{h}\frac{A_e}{A} U_e \Longrightarrow \overline{U_x} \cong \frac{2\,r}{h}\, U_b = \frac{2\,r}{h}\,\frac{dh}{dt} \tag{7.20}$$

In this analysis, for simplicity, this quantity will be used.

The averaged velocity in the $y$ direction is zero because the flow is symmetrical[7]. However, the change of the kinetic energy due to the change in the velocity field isn't zero. The kinetic energy of the tank or container is based on the half part as shown in Figure 7.3. Similar estimate that was done for $x$ direction can be done to every side of the opening if they are not symmetrical. Since in this case the geometry is assumed to be symmetrical one side is sufficient as

$$\overline{U_y} \cong \frac{(\pi - 2)r}{8\,h}\,\frac{dh}{dt} \tag{7.21}$$

[6]This assumption is appropriated only under certain conditions which include the geometry of the tank or container and the liquid properties. A discussion about this issue will be presented in the Dimensional Chapter and is out of the scope of this chapter. Also note that the straight surface assumption is not the same surface tension effects zero.

Also notice that the surface velocity is not zero. The surface has three velocity components which non have them vanish. However, in this discussion it is assumed that surface has only one component in $z$ direction. Hence it requires that velocity profile in $x$ $y$ to be parabolic. Second reason for this exercise the surface velocity has only one component is to avoid dealing with Bar-Meir's instability.

[7]For the mass conservation analysis, the velocity is zero for symmetrical geometry and some other geometries. However, for the energy analysis the averaged velocity cannot be considered zero.

The energy balance can be expressed by equation (7.15) which is applicable to this case. The temperature is constant[8]. In this light, the following approximation can be written

$$\dot{Q} = \frac{E_u}{dt} = h_{in} - h_{out} = 0 \qquad (7.22)$$

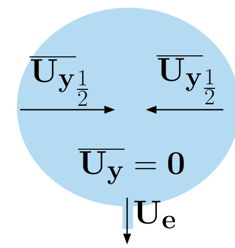



*Fig. -7.3. How to compensate and estimate the kinetic energy when averaged Velocity is zero.*

The boundary shear work is zero because the velocity at tank boundary or walls is zero. Furthermore, the shear stresses at the exit are normal to the flow direction hence the shear work is vanished. At the free surface the velocity has only normal component[9] and thus shear work vanishes there as well. Additionally, the internal shear work is assumed negligible.

$$\dot{W}_{shear} = \dot{W}_{shaft} = 0 \qquad (7.23)$$

Now the energy equation deals with no "external" effects. Note that the (exit) velocity on the upper surface is zero $U_{rn} = 0$.

Combining all these information results in

$$\overbrace{\frac{d}{dt}\int_V \left(\frac{U^2}{2} + g\,z\right)\rho\, dV}^{\text{internal energy change}} + \overbrace{\overbrace{\int_A \left(\frac{P_e}{\rho} + \frac{{U_e}^2}{2}\right) U_e\,\rho\, dA}^{\text{energy in and out}} - \overbrace{\int_A P_a\, U_b\, dA}^{\text{upper surface work}}}^{\text{energy flow out}} = 0 \qquad (7.24)$$

Where $U_b$ is the upper boundary velocity, $P_a$ is the external pressure and $P_e$ is the exit pressure[10].

The pressure terms in equation (7.24) are

$$\int_A \frac{P_e}{\rho} U_e\, \rho dA - \int_A P_a\, U_b\, dA = P_e \int_A U_e\, dA - P_a \int_A U_b\, dA \qquad (7.25)$$

It can be noticed that $P_a = P_e$ hence

$$P_a \overbrace{\left(\int_A U_e\, dA - \int_A U_b\, dA\right)}^{=0} = 0 \qquad (7.26)$$

[8]This approach is a common approximation. Yet, why this approach is correct in most cases is not explained here. Clearly, the dissipation creates a loss that has temperature component. In this case, this change is a function of Eckert number, $Ec$ which is very small. The dissipation can be neglected for small $Ec$ number. $Ec$ number is named after this author's adviser, E.R.G. Eckert. A discussion about this effect will be presented in the dimensional analysis chapter. Some examples how to calculate these losses will be resent later on.

[9]It is only the same assumption discussed earlier.

[10]It is assumed that the pressure in exit across section is uniform and equal surroundings pressure.

The governing equation (7.24) is reduced to

$$\frac{d}{dt}\int_V \left(\frac{U^2}{2} + g\,z\right)\rho\,dV - \int_A \left(\frac{{U_e}^2}{2}\right) U_e\,\rho\,dA = 0 \tag{7.27}$$

The minus sign is because the flow is out of the control volume.

Similarly to the previous chapter which the integral momentum will be replaced by some kind of average. The terms under the time derivative can be divided into two terms as

$$\frac{d}{dt}\int_V \left(\frac{U^2}{2} + g\,z\right)\rho dV = \frac{d}{dt}\int_V \frac{U^2}{2}\,dV + \frac{d}{dt}\int_V g\,z\,\rho\,dV \tag{7.28}$$

The second integral (in the r.h.s) of equation (7.28) is

$$\frac{d}{dt}\int_V g\,z\,\rho\,dV = g\,\rho\,\frac{d}{dt}\int_A\int_0^h z\,\overbrace{dz\,dA}^{dV} \tag{7.29}$$

Where $h$ is the height or the distance from the surface to exit. The inside integral can be evaluated as

$$\int_0^h z dz = \frac{h^2}{2} \tag{7.30}$$

Substituting the results of equation (7.30) into equation (7.29) yields

$$g\,\rho\,\frac{d}{dt}\int_A \frac{h^2}{2}\,dA = g\,\rho\,\frac{d}{dt}\left(\frac{h}{2}\,\overbrace{h\,A}^{V}\right) = g\,\rho\,A\,h\,\frac{d\,h}{dt} \tag{7.31}$$

The kinetic energy related to the averaged velocity with a correction factor which depends on the geometry and the velocity profile. Furthermore, Even the averaged velocity is zero the kinetic energy is not zero and another method should be used.

A discussion on the correction factor is presented to provide a better "averaged" velocity. A comparison between the actual kinetic energy and the kinetic energy due to the "averaged" velocity (to be called the averaged kinetic energy) provides a correction coefficient. The first integral can be estimated by examining the velocity profile effects. The averaged velocity is

$$U_{ave} = \frac{1}{V}\int_V U dV \tag{7.32}$$

The total kinetic energy for the averaged velocity is

$$\rho\,{U_{ave}}^2\,V = \rho\left(\frac{1}{V}\int_V U dV\right)^2 V = \rho\left(\int_V U dV\right)^2 \tag{7.33}$$

The general correction factor is the ratio of the above value to the actual kinetic energy as

$$C_F = \frac{\left(\displaystyle\int_V \rho\, U\, dV\right)^2}{\displaystyle\int_V \rho\, U^2\, dV} \neq \frac{\cancel{\rho}\,(U_{ave})^2\; V}{\displaystyle\int_V \cancel{\rho}\, U^2\, dV} \tag{7.34}$$

Here, $C_F$ is the correction coefficient. Note, the inequality sign because the density distribution for compressible fluid. The correction factor for a constant density fluid is

$$C_F = \frac{\left(\displaystyle\int_V \rho\, U\, dV\right)^2}{\displaystyle\int_V \rho\, U^2\, dV} = \frac{\left(\cancel{\rho}\displaystyle\int_V U\, dV\right)^2}{\cancel{\rho}\displaystyle\int_V U^2\, dV} = \frac{{U_{ave}}^2\, V}{\displaystyle\int_V U^2\, dV} \tag{7.35}$$

This integral can be evaluated for any given velocity profile. A large family of velocity profiles is laminar or parabolic (for one directional flow)[11]. For a pipe geometry, the velocity is

$$U\left(\frac{r}{R}\right) = U(\bar{r}) = U_{max}\left(1-\bar{r}^2\right) = 2\,U_{ave}\left(1-\bar{r}^2\right) \tag{7.36}$$

It can be noticed that the velocity is presented as a function of the reduced radius[12]. The relationship between $U_{max}$ to the averaged velocity, $U_{ave}$ is obtained by using equation (7.32) which yields $1/2$.

Substituting equation (7.36) into equation (7.35) results

$$\frac{{U_{ave}}^2\, V}{\displaystyle\int_V U^2\, dV} = \frac{{U_{ave}}^2\, V}{\displaystyle\int_V \left(2\,U_{ave}\left(1-\bar{r}^2\right)\right)^2\, dV} = \frac{{U_{ave}}^2\, V}{\dfrac{4\,{U_{ave}}^2\,\pi\, L\, R^2}{3}} = \frac{3}{4} \tag{7.37}$$

The correction factor for many other velocity profiles and other geometries can be smaller or larger than this value. For circular shape, a good guess number is about 1.1. In this case, for simplicity reason, it is assumed that the averaged velocity indeed represent the energy in the tank or container. Calculations according to this point can improve the accurately based on the above discussion.

> The difference between the "averaged momentum" velocity and the "averaged kinetic" velocity is also due to the fact that energy is added for different directions while in the momentum case, different directions cancel each other out.

[11] Laminar flow is not necessarily implies that the flow velocity profile is parabolic. The flow is parabolic only when the flow is driven by pressure or gravity. More about this issue in the Differential Analysis Chapter.

[12] The advantage is described in the Dimensional Analysis Chapter.

The unsteady state term then obtains the form

$$\frac{d}{dt}\int_V \rho \left(\frac{U^2}{2} + g\,y\right) dV \quad \cong \rho\,\frac{d}{dt}\left(\left[\frac{\overline{U}^2}{2} + \frac{g\,h}{2}\right]\overbrace{h\,A}^{V}\right) \tag{7.38}$$

The relationship between the boundary velocity to the height (by definition) is

$$U_b = \frac{dh}{dt} \tag{7.39}$$

Therefore, the velocity in the $z$ direction[13] is

$$U_z = \frac{dh}{dt} \tag{7.40}$$

$$U_e = \frac{A}{A_e}\frac{dh}{dt} = -U_b\frac{dh}{dt} \tag{7.41}$$

Combining all the three components of the velocity (Pythagorean Theorem) as

$$\overline{U}^2 \cong \overline{U_x}^2 + \overline{U_y}^2 + \overline{U_z}^2 \tag{7.42}$$

$$\overline{U}^2 \cong \left(\frac{(\pi-2)\,r}{8\,h}\frac{dh}{dt}\right)^2 + \left(\frac{(\pi-1)\,r}{4\,h}\frac{dh}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2 \tag{7.43}$$

$$\overline{U} \cong \frac{dh}{dt}\,\overbrace{\sqrt{\left(\frac{(\pi-2)\,r}{8\,h}\right)^2 + \left(\frac{(\pi-1)\,r}{4\,h}\right)^2 + 1^2}}^{f(G)} \tag{7.44}$$

It can be noticed that $f(G)$ is a weak function of the height inverse. Analytical solution of the governing equation is possible including this effect of the height. However, the mathematical complication are enormous[14] and this effect is assumed negligible and the function to be constant.

[13] A similar point was provided in mass conservation Chapter 5. However, it easy can be proved by construction the same control volume. The reader is encouraged to do it to get acquainted with this concept.

[14] The solution, not the derivation, is about one page. It must be remembered that is effect extremely important in the later stages of the emptying of the tank. But in the same vain, some other effects have to be taken into account which were neglected in construction of this model such as upper surface shape.

The last term is

$$\int_A \frac{{U_e}^2}{2} U_e\,\rho\,dA = \frac{{U_e}^2}{2} U_e\,\rho\,A_e = \frac{1}{2}\left(\frac{dh}{dt}\frac{A}{A_e}\right)^2 U_e\,\rho\,A_e \tag{7.45}$$

Combining all the terms into equation (7.27) results in

$$\cancel{\rho}\,\frac{d}{dt}\left(\left[\frac{\overline{U}^2}{2}+\frac{g\,h}{2}\right]\overbrace{h\,A}^{V}\right) - \frac{1}{2}\left(\frac{dh}{dt}\right)^2\left(\frac{A}{A_e}\right)^2 U_e\,\cancel{\rho}\,A_e = 0 \tag{7.46}$$

taking the derivative of first term on l.h.s. results in

$$\frac{d}{dt}\left[\frac{\overline{U}^2}{2}+\frac{g\,h}{2}\right] h\,A + \left[\frac{\overline{U}^2}{2}+\frac{g\,h}{2}\right] A\,\frac{dh}{dt} - \frac{1}{2}\left(\frac{dh}{dt}\right)^2\left(\frac{A}{A_e}\right)^2 U_e\,A_e = 0 \tag{7.47}$$

Equation (7.47) can be rearranged and simplified and combined with mass conservation [15].

*Advance material can be skipped*

Dividing equation (7.46) by $U_e\,A_e$ and utilizing equation (7.40)

$$\frac{d}{dt}\left[\frac{\overline{U}^2}{2}+\frac{g\,h}{2}\right]\frac{h\,A}{U_e\,A_e} + \left[\frac{\overline{U}^2}{2}+\frac{g\,h}{2}\right]\cancel{\overbrace{A\,\frac{dh}{dt}}^{A\,\frac{A_e}{A}\,U_e}} - \frac{1}{2}\left(\frac{dh}{dt}\right)^2\left(\frac{A}{A_e}\right)^2 \cancel{U_e\,A_e} = 0 \tag{7.48}$$

Notice that $\overline{U} = U_b\,f(G)$ and thus

$$\overbrace{\overline{U}}^{f(G)\,U_b}\,\frac{d\overline{U}}{dt}\,\frac{h\,A}{U_e\,A_e} + \frac{g}{2}\,\frac{dh}{dt}\,\frac{h\,A}{U_e\,A_e} + \left[\frac{\overline{U}^2}{2}+\frac{g\,h}{2}\right] - \frac{1}{2}\left(\frac{dh}{dt}\right)^2\left(\frac{A}{A_e}\right)^2 = 0 \tag{7.49}$$

Further rearranging to eliminate the "flow rate" transforms to

$$f(G)\,h\,\frac{d\overline{U}}{dt}\overbrace{\cancel{\left(\frac{U_b\,A}{U_e\,A_e}\right)}}^{1} + \frac{g\,h}{2}\,\overbrace{\cancel{\frac{\frac{dh}{dt}\,A}{U_e\,A_e}}}^{1} + \left[\frac{f(G)^2}{2}\left(\frac{dh}{dt}\right)^2 + \frac{g\,h}{2}\right] - \frac{1}{2}\left(\frac{dh}{dt}\right)^2\left(\frac{A}{A_e}\right)^2 = 0 \tag{7.50}$$

$$f(G)^2\,h\,\frac{d^2h}{dt^2} + \frac{g\,h}{2} + \left[\frac{f(G)^2}{2}\left(\frac{dh}{dt}\right)^2 + \frac{g\,h}{2}\right] - \frac{1}{2}\left(\frac{dh}{dt}\right)^2\left(\frac{A}{A_e}\right)^2 = 0 \tag{7.51}$$

[15] This part can be skipped to end of "advanced material".

*End Advance material*

Combining the $gh$ terms into one yields

$$f(G)^2\, h\, \frac{d^2h}{dt^2} + g\, h + \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \left[f(G)^2 - \left(\frac{A}{A_e}\right)^2\right] = 0 \tag{7.52}$$

Defining a new tank emptying parameter, $T_e$, as

$$T_e = \left(\frac{A}{f(G)\, A_e}\right)^2 \tag{7.53}$$

This parameter represents the characteristics of the tank which controls the emptying process. Dividing equation (7.52) by $f(G)^2$ and using this parameter, equation (7.52) after minor rearrangement transformed to

$$h\left(\frac{d^2h}{dt^2} + \frac{g\, {A_e}^2}{T_e\, A^2}\right) + \frac{1}{2}\left(\frac{dh}{dt}\right)^2 [1 - T_e] = 0 \tag{7.54}$$

The solution can either of these equations[16]

$$-\int \frac{dh}{\sqrt{\dfrac{(k_1\, T_e - 2\, k_1)\; e^{\ln(h)\, Te} + 2\, g\, h^2}{h\; (Te - 2)\; f(G)}}} = t + k_2 \tag{7.55}$$

or

$$\int \frac{dh}{\sqrt{\dfrac{(k_1\, T_e - 2\, k_1)\; e^{\ln(h)\, Te} + 2\, g\, h^2}{h\; (Te - 2)\; f(G)}}} = t + k_2 \tag{7.56}$$

The solution with the positive solution has no physical meaning because the height cannot increase with time. Thus define function of the height as

$$f(h) = -\int \frac{dh}{\sqrt{\dfrac{(k_1\, T_e - 2\, k_1)\; e^{\ln(h)\, Te} + 2\, g\, h^2}{h\; (Te - 2)\; f(G)}}} \tag{7.57}$$

The initial condition for this case are: one the height initial is

$$h(0) = h_0 \tag{7.58}$$

[16] A discussion about this equation appear in the mathematical appendix.

The initial boundary velocity is

$$\frac{dh}{dt} = 0 \tag{7.59}$$

This condition pose a physical limitation[17] which will be ignored. The first condition yields

$$k_2 = -f(h_0) \tag{7.60}$$

The second condition provides

$$\frac{dh}{dt} = 0 = \sqrt{\frac{(k_1\,T_e - 2\,k_1)\;e^{\ln(h_0)\,Te} + 2\,g\,{h_0}^2}{h_0\;(Te-2)\;f(G)}} \tag{7.61}$$

The complication of the above solution suggest a simplification in which

$$\frac{d^2h}{dt^2} << \frac{g\,{A_e}^2}{T_e\,A^2} \tag{7.62}$$

which reduces equation (7.54) into

$$h\left(\frac{g\,{A_e}^2}{T_e\,A^2}\right) + \frac{1}{2}\left(\frac{dh}{dt}\right)^2[1-T_e] = 0 \tag{7.63}$$

While equation (7.63) is still non linear equation, the non linear element can be removed by taking negative branch (height reduction) of the equation as

$$\left(\frac{dh}{dt}\right)^2 = \frac{2\,g\,h}{-1+\left(\frac{A}{A_e}\right)^2} \tag{7.64}$$

It can be noticed that $T_e$ "disappeared" from the equation. And taking the "positive" branch

$$\frac{dh}{dt} = \frac{\sqrt{2\,g\,h}}{\sqrt{1-\left(\frac{A}{A_e}\right)^2}} \tag{7.65}$$

The nature of first order Ordinary Differential Equation that they allow only one initial condition. This initial condition is the initial height of the liquid. The initial velocity field was eliminated by the approximation (remove the acceleration term). Thus it is assumed that the initial velocity is not relevant at the core of the process at hand. It is

[17]For the initial condition speed of sound has to be taken into account. Thus for a very short time, the information about opening of the valve did not reached to the surface. This information travel in characteristic sound speed which is over 1000 $m/sec$. However, if this phenomenon is ignored this solution is correct.

correct only for large ratio of $h/r$ and the error became very substantial for small value of $h/r$.

Equation (7.65) integrated to yield

$$\left(1-\left(\frac{A}{A_e}\right)^2\right)\int_{h_0}^{h}\frac{dh}{\sqrt{2\,g\,h}}=\int_0^t dt \tag{7.66}$$

The initial condition has been inserted into the integral which its solution is

$$\left(1-\left(\frac{A}{A_e}\right)^2\right)\frac{h-h_0}{\sqrt{2\,g\,h}}=t \tag{7.67}$$

$$U_e=\frac{dh}{dt}\frac{A}{A_e}=\frac{\sqrt{2\,g\,h}}{\sqrt{1-\left(\frac{A}{A_e}\right)^2}}\frac{A}{A_e}=\frac{\sqrt{2\,g\,h}}{\sqrt{1-\left(\frac{A_e}{A}\right)^2}} \tag{7.68}$$

If the area ratio $A_e/A << 1$ then

$$U\cong\sqrt{2\,g\,h} \tag{7.69}$$

Equation (7.69) is referred in the literature as Torricelli's equation[18]

This analysis has several drawbacks which limits the accuracy of the calculations. Yet, this analysis demonstrates the usefulness of the integral analysis to provide a reasonable solution. This analysis can be improved by experimental investigating the phenomenon. The experimental coefficient can be added to account for the dissipation and other effects such

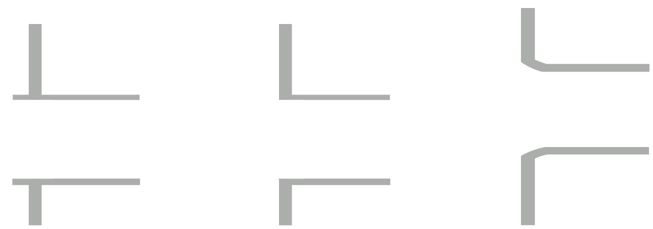
$$\frac{dh}{dt}\cong C\,\sqrt{2\,g\,h} \tag{7.70}$$

The loss coefficient can be expressed as

$$C=Kf\left(\frac{U^2}{2}\right) \tag{7.71}$$

A few loss coefficients for different configuration is given following Figure 7.4.

[18]Evangelista Torricelli (October 15, 1608 October 25, 1647) was an Italian physicist and mathematician. He derived this equation based on similar principle to Bernoulli equation (which later leads to Bernoulli's equation). Today the exact reference to his work is lost and only "sketches" of his lecture elude work. He was student (not formal) and follower of Galileo Galilei. It seems that Torricelli was an honest man who gave to others and he died at young age of 39 while in his prime.



(a) Projecting pipe K=1.

(b) Sharp edge pipe connection K=0.5.

(c) Rounded inlet pipe K=0.04.

*Fig. -7.4. Typical resistance for selected outlet configuration.*

## 7.2 Limitation of Integral Approach

Some of accuracy issues to enhance the quality and improvements of the integral method were suggested in the analysis of the emptying tank. There are problems that the integral methods even with these enhancements simply cannot tackle.

The improvements to the integral methods are the corrections to the estimates of the energy or other quantities in the conservation equations. In the calculations of the exit velocity of a tank, two such corrections were presented. The first type is the prediction of the velocities profile (or the concentration profile). The second type of corrections is the understanding that averaged of the total field is different from the averaged of different zooms. In the case of the tank, the averaged velocity in $x$ direction is zero yet the averaged velocity in the two zooms (two halves) is not zero. In fact, the averaged energy in the $x$ direction contributes or effects the energy equation. The accuracy issues that integral methods intrinsically suffers from no ability to exact flow field and thus lost the accuracy as was discussed in the example. The integral method does not handle the problems such as the free surface with reasonable accuracy. Furthermore, the knowledge of whether the flow is laminar or turbulent (later on this issue) has to come from different techniques. Hence the prediction can skew the actual predictions.

In the analysis of the tank it was assumed that the dissipation can be ignored. In cases that dissipation play major role, the integral does not provide a sufficient tool to analyze the issue at hand. For example, the analysis of the oscillating manometer cannot be carried by the integral methods. A liquid in manometer is disturbed from a rest by a distance of $H_0$. The description $H(t)$ as a function of time requires exact knowledge of the velocity field. Additionally, the integral methods is

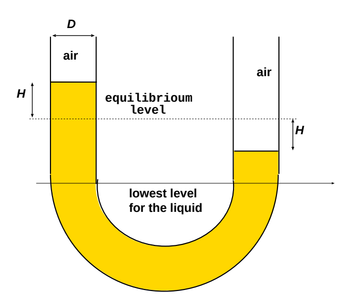



*Fig. -7.5. Flow in an oscillating manometer.*

too crude to handle issues of free interface.
These problem were minor for the emptying the tank but for the oscillating manometer it is the core of the problem. Hence different techniques are required.

The discussion on the limitations was not provided to discard usage of this method but rather to provide a guidance of use with caution. The integral method is a powerful and yet simple method but has has to be used with the limitations of the method in mind.

## 7.3 Approximation of Energy Equation

The emptying the tank problem was complicated even with all the simplifications that were carried. Engineers in order to reduce the work further simplify the energy equation. It turn out that these simplifications can provide reasonable results and key understanding of the physical phenomena and yet with less work, the problems can be solved. The following sections provides further explanation.

### 7.3.1 Energy Equation in Steady State

The steady state situation provides several ways to reduce the complexity. The time derivative term can be eliminated since the time derivative is zero. The acceleration term must be eliminated for the obvious reason. Hence the energy equation is reduced to

Steady State Equation

$$\dot{Q} - \dot{W}_{shear} - \dot{W}_{shaft} = \int_S \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\,dA + \int_S P U_{bn} dA \tag{7.72}$$

If the flow is uniform or can be estimated as uniform, equation (7.72) is reduced to

Steady State Equation & uniform

$$\dot{Q} - \dot{W}_{shear} - \dot{W}_{shaft} = \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho A_{out} - \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho A_{in} + P U_{bn} A_{out} - P U_{bn} A_{in} \tag{7.73}$$

It can be noticed that last term in equation (7.73) for non-deformable control volume does not vanished. The reason is that while the velocity is constant, the pressure is different. For a stationary fix control volume the energy equation, under this simplification transformed to

$$\dot{Q} - \dot{W}_{shear} - \dot{W}_{shaft} = \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho A_{out} - \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho A_{in} \tag{7.74}$$

Dividing equation the mass flow rate provides

Steady State Equation, Fix $\dot{m}$ & uniform

$$\dot{q} - \dot{w}_{shear} - \dot{w}_{shaft} = \left(h + \frac{U^2}{2} + g\,z\right)\bigg|_{out} - \left(h + \frac{U^2}{2} + g\,z\right)\bigg|_{in} \tag{7.75}$$

### 7.3.2 Energy Equation in Frictionless Flow and Steady State

In cases where the flow can be estimated without friction or where a quick solution is needed the friction and other losses are illuminated from the calculations. This imaginary fluid reduces the amount of work in the calculations and Ideal Flow Chapter is dedicated in this book. The second low is the core of "no losses" and can be employed when calculations of this sort information is needed. Equation (2.21) which can be written as

$$dq_{rev} = T\,ds = dE_u + P\,dv \tag{7.76}$$

Using the multiplication rule change equation (7.76)

$$dq_{rev} = dE_u + d\,(P\,v) - v\,dP = dE_u + d\left(\frac{P}{\rho}\right) - v\,dP \tag{7.77}$$

integrating equation (7.77) yields

$$\int dq_{rev} = \int dE_u + \int d\left(\frac{P}{\rho}\right) - \int v\,dP \tag{7.78}$$

$$q_{rev} = E_u + \left(\frac{P}{\rho}\right) - \int \frac{dP}{\rho} \tag{7.79}$$

Integration over the entire system results in

$$Q_{rev} = \int_V \overbrace{\left(E_u + \left(\frac{P}{\rho}\right)\right)}^{h} \rho\,dV - \int_V \left(\int \frac{dP}{\rho}\right) \rho\,dV \tag{7.80}$$

Taking time derivative of the equation (7.80) becomes

$$\dot{Q}_{rev} = \frac{D}{Dt}\int_V \overbrace{\left(E_u + \left(\frac{P}{\rho}\right)\right)}^{h} \rho\,dV - \frac{D}{Dt}\int_V \left(\int \frac{dP}{\rho}\right) \rho\,dV \tag{7.81}$$

Using the Reynolds Transport Theorem to transport equation to control volume results in

$$\dot{Q}_{rev} = \frac{d}{dt}\int_V h\,\rho\,dV + \int_A h\,U_{rn}\,\rho\,dA + \frac{D}{Dt}\int_V \left(\int \frac{dP}{\rho}\right) \rho\,dV \tag{7.82}$$

As before equation (7.81) can be simplified for uniform flow as

$$\dot{Q}_{rev} = \dot{m}\left[(h_{out} - h_{in}) - \left(\int \frac{dP}{\rho}\bigg|_{out} - \int \frac{dP}{\rho}\bigg|_{in}\right)\right] \tag{7.83}$$

or

$$\dot{q}_{rev} = (h_{out} - h_{in}) - \left(\int \frac{dP}{\rho}\bigg|_{out} - \int \frac{dP}{\rho}\bigg|_{in}\right) \tag{7.84}$$

Subtracting equation (7.84) from equation (7.75) results in

$$0 = w_{shaft} + \overbrace{\left(\int \frac{dP}{\rho}\bigg|_{2} - \int \frac{dP}{\rho}\bigg|_{1}\right)}^{\text{change in pressure energy}} + \overbrace{\frac{{U_2}^2 - {U_1}^2}{2}}^{\text{change in kinetic energy}} + \overbrace{g\,(z_2 - z_1)}^{\text{change in potential energy}} \tag{7.85}$$

Equation (7.85) for constant density is

$$0 = w_{shaft} + \frac{P_2 - P_1}{\rho} + \frac{{U_2}^2 - {U_1}^2}{2} + g\,(z_2 - z_1) \tag{7.86}$$

For no shaft work equation (7.86) reduced to

$$0 = \frac{P_2 - P_1}{\rho} + \frac{{U_2}^2 - {U_1}^2}{2} + g\,(z_2 - z_1) \tag{7.87}$$

## 7.4 Energy Equation in Accelerated System

In the discussion so far, it was assumed that the control volume is at rest. The only acceptation to the above statement, is the gravity that was compensated by the gravity potential. In building the gravity potential it was assumed that the gravity is a conservative force. It was pointed earlier in this book that accelerated forces can be translated to potential force. In many cases, the control volume is moving in accelerated coordinates. These accelerations will be translated to potential energy.

The accelerations are referring to two kinds of acceleration, linear and rotational. There is no conceptional difference between these two accelerations. However, the mathematical treatment is somewhat different which is the reason for the separation. General Acceleration can be broken into a linear acceleration and a rotating acceleration.

### 7.4.1 Energy in Linear Acceleration Coordinate

The potential is defined as

$$P.E. = -\int_{ref}^{2} \boldsymbol{F} \cdot \boldsymbol{d\ell} \tag{7.88}$$

In Chapter 3 a discussion about gravitational energy potential was presented. For example, for the gravity force is

$$F = -\frac{G\,M\,m}{r^2} \tag{7.89}$$

Where $G$ is the gravity coefficient and $M$ is the mass of the Earth. $r$ and $m$ are the distance and mass respectively. The gravity potential is then

$$PE_{gravity} = -\int_{\infty}^{r} -\frac{G\,M\,m}{r^2} dr \tag{7.90}$$

The reference was set to infinity. The gravity force for fluid element in small distance then is $g\,dz\,dm$. The work this element moving from point 1 to point 2 is

$$\int_1^2 g\,dz\,dm = g\,(z_2 - z_1)\,dm \tag{7.91}$$

The total work or potential is the integral over the whole mass.

### 7.4.2 Linear Accelerated System

The acceleration can be employed in similar fashion as the gravity force. The linear acceleration "creates" a conservative force of constant force and direction. The "potential" of moving the mass in the field provides the energy. The Force due to the acceleration of the field can be broken into three coordinates. Thus, the element of the potential is

$$d\,PE_a = \boldsymbol{a} \cdot d\boldsymbol{\ell}\,dm \tag{7.92}$$

The total potential for element material

$$PE_a = \int_{(0)}^{(1)} \boldsymbol{a} \cdot d\boldsymbol{\ell}\,dm = \left(a_x\,(x_1 - x_0)\,a_y\,(y_1 - y_0)\,a_z\,(z_1 - z_0)\right)\,dm \tag{7.93}$$

At the origin (of the coordinates) $x = 0$, $y = 0$, and $z = 0$. Using this trick the notion of the $a_x\,(x_1 - x_0)$ can be replaced by $a_x\,x$. The same can be done for the other two coordinates. The potential of unit material is

$$PE_{a\,total} = \int_{sys} \left(a_x\,x + a_y\,y + a_z\,z\right)\,\rho\,dV \tag{7.94}$$

The change of the potential with time is

$$\frac{D}{Dt} PE_{a\,total} = \frac{D}{Dt} \int_{sys} \left(a_x\,x + a_y\,y + a_z\,z\right)\,dm \tag{7.95}$$

Equation can be added to the energy equation as

$$\dot{Q} - \dot{W} = \frac{D}{Dt} \int_{sys} \left[ E_u + \frac{U^2}{2} + a_x\, x + a_y\, y + (a_z + g) z \right] \rho\, dV \tag{7.96}$$

The Reynolds Transport Theorem is used to transferred the calculations to control volume as

Energy Equation in Linear Accelerated Coordinate

$$\begin{aligned} \dot{Q} - \dot{W} = \frac{d}{dt} \int_{cv} \left[ E_u + \frac{U^2}{2} + a_x\, x + a_y\, y + (a_z + g) z \right] \rho\, dV \\ + \int_{cv} \left( h + \frac{U^2}{2} + a_x\, x + a_y\, y + (a_z + g) z \right) U_{rn}\, \rho\, dA \\ + \int_{cv} P\, U_{bn}\, dA \end{aligned} \tag{7.97}$$

### 7.4.3 Energy Equation in Rotating Coordinate System

The coordinate system rotating around fix axises creates a similar conservative potential as a linear system. There are two kinds of acceleration due to this rotation; one is the centrifugal and one the Coriolis force. To understand it better, consider a particle which moves with the our rotating system. The forces acting on particles are

$$\boldsymbol{F} = \left( \overbrace{\omega^2\, r\, \hat{r}}^{centrifugal} + \overbrace{2\boldsymbol{U} \times \boldsymbol{\omega}}^{Coriolis} \right) dm \tag{7.98}$$

The work or the potential then is

$$PE = \left( \omega^2\, r\, \hat{r} + 2\boldsymbol{U} \times \boldsymbol{\omega} \right) \cdot d\boldsymbol{\ell}\, dm \tag{7.99}$$

The cylindrical coordinate are

$$d\boldsymbol{\ell} = dr\hat{r} + r\, d\theta\, \hat{\theta} + dz\, \hat{k} \tag{7.100}$$

where $\hat{r}$, $\hat{\theta}$, and $\hat{k}$ are units vector in the coordinates $r$, $\theta$ and $z$ respectively. The potential is then

$$PE = \left( \omega^2\, r\, \hat{r} + 2\boldsymbol{U} \times \boldsymbol{\omega} \right) \cdot \left( dr\hat{r} + r\, d\theta\, \hat{\theta} + dz\, \hat{k} \right) dm \tag{7.101}$$

The first term results in $\omega^2\, r^2$ (see for explanation in the appendix 307 for vector explanation). The cross product is zero of

$$\boldsymbol{U} \times \boldsymbol{\omega} \times \boldsymbol{U} = \boldsymbol{U} \times \boldsymbol{\omega} \times \boldsymbol{\omega} = 0$$

because the first multiplication is perpendicular to the last multiplication. The second part is

$$\left(2\boldsymbol{U}\times\boldsymbol{\omega}\right)\cdot d\boldsymbol{\ell}\,dm \tag{7.102}$$

This multiplication does not vanish with the exception of the direction of $\boldsymbol{U}$. However, the most important direction is the direction of the velocity. This multiplication creates lines (surfaces ) of constant values. From a physical point of view, the flux of this property is important only in the direction of the velocity. Hence, this term canceled and does not contribute to the potential.

The net change of the potential energy due to the centrifugal motion is

$$PE_{centrifugal} = -\int_1^2 \omega^2\, r^2\, dr\, dm = \frac{\omega^2\left({r_1}^2 - {r_2}^2\right)}{2}\, dm \tag{7.103}$$

Inserting the potential energy due to the centrifugal forces into the energy equation yields

Energy Equation in Accelerated Coordinate

$$\begin{aligned}\dot{Q} - \dot{W} = \frac{d}{dt}\int_{cv}\left[E_u + \frac{U^2}{2} + a_x\,x + a_y\,y + (a_z+g)z - \frac{\omega^2\,r^2}{2}\right]\rho\,dV \\ + \int_{cv}\left(h + \frac{U^2}{2} + a_x\,x + a_y\,y + (a_z+g) - z\,\frac{\omega^2\,r^2}{2}\right)U_{rn}\,\rho\,dA \\ + \int_{cv} P\,U_{bn}\,dA\end{aligned} \tag{7.104}$$

### 7.4.4 Simplified Energy Equation in Accelerated Coordinate

#### 7.4.4.1 Energy Equation in Accelerated Coordinate with Uniform Flow

One of the way to simplify the general equation (7.104) is to assume uniform flow. In that case the time derivative term vanishes and equation (7.104) can be written as

Energy Equation in steady state

$$\begin{aligned}\dot{Q} - \dot{W} = \int_{cv}\left(h + \frac{U^2}{2} + a_x\,x + a_y\,y + (a_z+g) - z\,\frac{\omega^2\,r^2}{2}\right)U_{rn}\,\rho\,dA \\ + \int_{cv} P\,U_{bn}\,dA\end{aligned} \tag{7.105}$$

Further simplification of equation (7.105) by assuming uniform flow for which

$$\dot{Q} - \dot{W} = \left( h + \frac{\overline{U}^2}{2} + a_x\, x + a_y\, y + (a_z + g) - z\, \frac{\omega^2\, r^2}{2} \right) \overline{U}_{rn}\, \rho\, dA \\ + \int_{cv} P\, \overline{U}_{bn}\, dA \tag{7.106}$$

Note that the acceleration also have to be averaged. The correction factors have to introduced into the equation to account for the energy averaged verse to averaged velocity (mass averaged). These factor make this equation with larger error and thus less effective tool in the engineering calculation.

### 7.4.5 Energy Losses in Incompressible Flow

In the previous sections discussion, it was assumed that there are no energy loss. However, these losses are very important for many real world application. And these losses have practical importance and have to be considered in engineering system. Hence writing equation (7.15) when the energy and the internal energy as a separate identity as

$$\begin{gathered}\dot{W}_{shaft} = \frac{d}{dt} \int_V \left( \frac{U^2}{2} + g\, z \right) \rho\, dV + \\ \int_A \left( \frac{P}{\rho} + \frac{U^2}{2} + g\, z \right) U_{rn}\, \rho\, dA + \int_A P U_{bn} dA + \\ \overbrace{\frac{d}{dt} \int_V E_u\, \rho\, dV + \int_A E_u\, U_{rn}\, \rho\, dA - \dot{Q} - \dot{W}_{shear}}^{\text{energy loss}}\end{gathered} \tag{7.107}$$

Equation (7.107) sometimes written as

$$\begin{gathered}\dot{W}_{shaft} = \frac{d}{dt} \int_V \left( \frac{U^2}{2} + g\, z \right) \rho\, dV + \\ \int_A \left( \frac{P}{\rho} + \frac{U^2}{2} + g\, z \right) U_{rn}\, \rho\, dA + \int_A P U_{bn} dA + \text{energy loss}\end{gathered} \tag{7.108}$$

Equation can be further simplified under assumption of uniform flow and steady state as

$$\dot{w}_{shaft} = \left( \frac{P}{\rho} + \frac{U^2}{2} + g\, z \right) \bigg|_{out} - \left( \frac{P}{\rho} + \frac{U^2}{2} + g\, z \right) \bigg|_{in} + \text{energy loss} \tag{7.109}$$

Equation (7.109) suggests that term $h + \frac{U^2}{2} + g\, z$ has a special meaning (because it remained constant under certain conditions). This term, as will be shown, has to be constant for frictionless flow without any addition and loss of energy. This term represents the "potential energy." The loss is the combination of the internal energy/enthalpy with heat transfer. For example, fluid flow in a pipe has resistance and energy dissipation. The dissipation is lost energy that is transferred to the surroundings. The loss is normally is a strong function of the velocity square, $U^2/2$. There are several categories of the loss which referred as minor loss (which are not minor), and duct losses. These losses will be tabulated later on.

If the energy loss is negligible and the shaft work vanished or does not exist equation (7.109) reduces to simple Bernoulli's equation.

Simple Bernoulli

$$0 = \left.\left(\frac{P}{\rho} + \frac{U^2}{2} + g\,z\right)\right|_{out} - \left.\left(\frac{P}{\rho} + \frac{U^2}{2} + g\,z\right)\right|_{in} \tag{7.110}$$

Equation (7.110) is only a simple form of Bernoulli's equation which was developed by Bernoulli's adviser, Euler. There also unsteady state and other form of this equation that will be discussed in differential equations Chapter.

## 7.5 Examples of Integral Energy Conservation

Example 7.1:

*Consider a flow in a long straight pipe. Initially the flow is in a rest. At time, $t_0$ the a constant pressure difference is applied on the pipe. Assume that flow is incompressible, and the resistance or energy loss is $f$. Furthermore assume that this loss is a function of the velocity square. Develop equation to describe the exit velocity as a function of time. State your assumptions.*

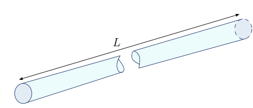



*Fig. -7.6. Flow in a long pipe when exposed to pressure difference.*

Solution

The mass balance on the liquid in the pipe results in

$$0 = \overbrace{\int_V \frac{\partial \rho}{\partial t} dV}^{=0} + \overbrace{\int_A \rho\, U_{bn} dA}^{=0} + \int_A \rho\, U_{rn} dA \Longrightarrow \cancel{\rho}\cancel{A} U_{in} = \cancel{\rho}\cancel{A} U_{exit} \tag{7.I.a}$$

There is no change in the liquid mass inside pipe and therefore the time derivative is zero (the same mass resides in the pipe at all time). The boundaries do not move and the second term is zero. Thus, the flow in and out are equal because the density is identical. Furthermore, the velocity is identical because the cross area is same.

It can be noticed that for the energy balance on the pipe, the time derivative can enter the integral because the control volume has fixed boundaries. Hence,

$$\dot{Q} - \overbrace{\dot{W}_{shear}}^{=0} + \overbrace{\dot{W}_{shaft}}^{=0} = \int_V \frac{d}{dt}\left(E_u + \frac{U^2}{2} + g\,z\right)\rho\,dV + \int_S \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\,dA + \int_S P U_{bn} dA \tag{7.I.b}$$

The boundaries shear work vanishes because the same arguments present before (the work, where velocity is zero, is zero. In the locations where the velocity does not vanished, such as in and out, the work is zero because shear stress are perpendicular to the velocity).

There is no shaft work and this term vanishes as well. The first term on the right hand side (with a constant density) is

$$\rho \int_{V_{pipe}} \frac{d}{dt}\left(E_u + \frac{U^2}{2} + \overbrace{g\,z}^{constant}\right) dV = \rho\,U\,\frac{d\,U}{dt}\overbrace{V_{pipe}}^{L\,\pi\,r^2} + \rho \int_{V_{pipe}} \frac{d}{dt}\left(E_u\right) dV \tag{7.I.c}$$

where $L$ is the pipe length, $r$ is the pipe radius, $U$ averaged velocity.

In this analysis, it is assumed that the pipe is perpendicular to the gravity line and thus the gravity is constant. The gravity in the first term and all other terms, related to the pipe, vanish again because the value of $z$ is constant. Also, as can be noticed from equation (7.I.a), the velocity is identical (in and out). Hence the second term becomes

$$\int_A \left(h + \left(\cancel{\frac{U^2}{2} + g\,z}\right)^{constant}\right)\rho\,U_{rn} dA = \int_A \overbrace{\left(E_u + \frac{P}{\rho}\right)}^{h} \rho\,U_{rn} dA \tag{7.I.d}$$

Equation (7.I.d) can be further simplified (since the area and averaged velocity are constant, additionally notice that $U = U_{rn}$) as

$$\int_A \left(E_u + \frac{P}{\rho}\right)\rho\,U_{rn} dA = \Delta P\,U\,A + \int_A \rho\,E_u\,U_{rn}\,dA \tag{7.I.e}$$

The third term vanishes because the boundaries velocities are zero and therefore

$$\int_A P\;U_{bn} dA = 0 \tag{7.I.f}$$

Combining all the terms results in

$$\dot{Q} = \rho\,U\,\frac{d\,U}{dt}\overbrace{V_{pipe}}^{L\,\pi\,r^2} + \rho\,\frac{d}{dt}\int_{V_{pipe}} E_u\,dV + \Delta P\,U\,dA + \int_A \rho\,E_u\,U\,dA \tag{7.I.g}$$

equation (7.I.g) can be rearranged as

$$\overbrace{\dot{Q} - \rho \int_{V_{pipe}} \frac{d\left(E_u\right)}{dt} dV - \int_A \rho\,E_u\,U\,dA}^{-K\frac{U^2}{2}} = \rho\,L\,\pi\,r^2\,U\,\frac{d\,U}{dt} + \left(P_{in} - P_{out}\right) U \tag{7.I.h}$$

The terms on the LHS (left hand side) can be combined. It common to assume (to view) that these terms are representing the energy loss and are a strong function of velocity square[19]. Thus, equation (7.I.h) can be written as

$$-K\,\frac{U^2}{2} = \rho\,L\,\pi\,r^2\,U\,\frac{d\,U}{dt} + (P_{in} - P_{out})\,U \tag{7.I.i}$$

Dividing equation (7.I.i) by $K\,U/2$ transforms equation (7.I.i) to

$$U + \frac{2\,\rho\,L\,\pi\,r^2}{K}\frac{d\,U}{dt} = \frac{2\,(P_{in} - P_{out})}{K} \tag{7.I.j}$$

Equation (7.I.j) is a first order differential equation. The solution this equation is described in the appendix and which is

$$U = e^{-\left(\frac{t\,K}{2\,\pi\,r^2\,\rho\,L}\right)}\left(\frac{2\,(P_{in} - P_{out})\,e^{\left(\frac{t\,K}{2\,\pi\,r^2\,\rho\,L}\right)}}{K} + c\right)e^{\left(\frac{2\,\pi\,r^2\,\rho\,t\,L}{K}\right)} \tag{7.I.k}$$

Applying the initial condition, $U(t = 0) = 0$ results in

$$U = \frac{2\,(P_{in} - P_{out})}{K}\left(1 - e^{-\left(\frac{t\,K}{2\,\pi\,r^2\,\rho\,L}\right)}\right) \tag{7.I.l}$$

The solution is an exponentially approaching the steady state solution. In steady state the flow equation (7.I.j) reduced to a simple linear equation. The solution of the linear equation and the steady state solution of the differential equation are the same.

$$U = \frac{2\,(P_{in} - P_{out})}{K} \tag{7.I.m}$$

Another note, in reality the resistance, $K$, is not constant but rather a strong function of velocity (and other parameters such as temperature[20], velocity range, velocity regime and etc.). This function will be discussed in a greater extent later on. Additionally, it should be noted that if momentum balance was used a similar solution (but not the same) was obtained (why? hint the difference of the losses accounted for).

End Solution

The following example combined the above discussion in the text with the above example (7.1).

[19] The shear work inside the liquid refers to molecular work (one molecule work on the other molecule). This shear work can be viewed also as one control volume work on the adjoined control volume.

[20] Via the viscosity effects.

Example 7.2:

*A large cylindrical tank with a diameter, $D$, contains liquid to height, $h$. A long pipe is connected to a tank from which the liquid is emptied. To analysis this situation, consider that the tank has a constant pressure above liquid (actually a better assumption of air with a constant mass.). The pipe is exposed to the surroundings and thus the pressure is $P_{atmos}$ at the pipe exit. Derive approximated equations that related the height in the large tank and the exit velocity at the pipe to pressure difference. Assume that the liquid is incompressible. Assume that the resistance or the friction in the pipe is a strong function to the velocity square in the tank. State all the assumptions that were made during the derivations.*

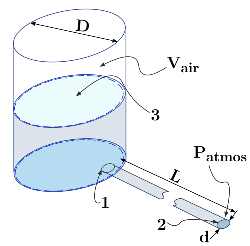



*Fig. -7.7. Liquid exiting a large tank trough a long tube.*

SOLUTION

This problem can split into two control volumes; one of the liquid in the tank and one of the liquid in pipe. Analysis of control volume in the tank was provided previously and thus needed to be sewed to Example 7.1. Note, the energy loss is considered (as opposed to the discussion in the text). The control volume in tank is depicted in Figure 7.7.

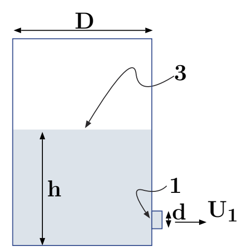



*Fig. -7.8. Tank control volume for Example 7.2.*

**Tank Control Volume**

The effect of the energy change in air side was neglected. The effect is negligible in most cases because air mass is small with exception the "spring" effect (expansion/compression effects). The mass conservation reads

$$\overbrace{\int_V \frac{\partial \rho}{\partial t} dV}^{=0} + \int_A \rho\, U_{bn} dA + \int_A \rho\, U_{rn} dA = 0 \tag{7.II.a}$$

The first term vanishes and the second and third terms remain and thus equation (7.II.a) reduces to

$$\cancel{\rho}\, U_1\, A_{pipe} = \cancel{\rho}\, U_3\, \overbrace{\pi\, R^2}^{A_{tank}} = \cancel{\rho}\, \frac{dh}{dt}\, \overbrace{\pi\, R^2}^{A_{tank}} \tag{7.II.b}$$

It can be noticed that $U_3 = dh/dt$ and $D = 2\,R$ and $d = 2\,r$ when the lower case refers to the pipe and the upper case referred to the tank. Equation (7.II.b) simply can be written when the area ratio is used (to be changed later if needed) as

$$U_1 \, A_{pipe} = \frac{dh}{dt} \, A_{tank} \Longrightarrow U_1 = \left(\frac{R}{r}\right)^2 \frac{dh}{dt} \tag{7.II.c}$$

The boundaries shear work and the shaft work are assumed to be vanished in the tank. Therefore, the energy conservation in the tank reduces to

$$\begin{gathered}\dot{Q} - \overbrace{\dot{W}_{shear}}^{=0} + \overbrace{\dot{W}_{shaft}}^{=0} = \frac{d}{dt} \int_{V_t} \left(E_u + \frac{{U_t}^2}{2} + g\,z\right) \rho \, dV + \\ \int_{A_1} \left(h + \frac{{U_t}^2}{2} + g\,z\right) U_{rn} \, \rho \, dA + \int_{A_3} P U_{bn} dA\end{gathered} \tag{7.II.d}$$

Where $U_t$ denotes the (the upper surface) liquid velocity of the tank. Moving all internal energy terms and the energy transfer to the right hand side of equation (7.II.d) to become

$$\begin{gathered}\frac{d}{dt} \int_{V_t} \left(\frac{{U_t}^2}{2} + g\,z\right) \rho \, dV + \int_{A_1} \left(\frac{P}{\rho} + \frac{{U_t}^2}{2} + g\,z\right) \overbrace{U_{rn}}^{U_1} \rho \, dA + \\ \int_{A_3} P \, \overbrace{U_{bn}}^{U_3} dA = \overbrace{\frac{d}{dt} \int_{V_t} E_u \rho \, dV + \int_{A_1} E_u \, \rho \, U_{rn} \, dA - \dot{Q}}^{K \, \frac{{U_t}^2}{2}}\end{gathered} \tag{7.111}$$

Similar arguments to those that were used in the previous discussion are applicable to this case. Using equation (7.38), the first term changes to

$$\frac{d}{dt} \int_V \rho \left(\frac{U^2}{2} + g\,z\right) dV \quad \cong \rho \, \frac{d}{dt} \left( \left[ \frac{\overline{U_t}^2}{2} + \frac{g\,h}{2} \right] \overbrace{h\,A}^{V} \right) \tag{7.II.e}$$

Where the velocity is given by equation (7.44). That is, the velocity is a derivative of the height with a correction factor, $U = dh/dt \times f(G)$. Since the focus in this book is primarily on the physics, $f(G) \equiv 1$ will be assumed. The pressure component of the second term is

$$\int_A \frac{P}{\cancel{\rho}} \, U_{rn} \, \cancel{\rho} \, dA = \rho \, P_1 \, U_1 \, A_1 \tag{7.II.f}$$

It is assumed that the exit velocity can be averaged (neglecting the velocity distribution effects). The second term can be recognized as similar to those by equation (7.45). Hence, the second term is

$$\int_A \left(\frac{U^2}{2} + \overbrace{g\,z}^{z=0}\right) U_{rn} \, \rho \, dA \cong \frac{1}{2} \left(\frac{dh}{dt} \frac{A_3}{A_1}\right)^2 U_1 \, \rho \, A_1 = \frac{1}{2} \left(\frac{dh}{dt} \frac{R}{r}\right)^2 U_1 \, \rho \, A_1 \tag{7.II.g}$$

The last term on the left hand side is

$$\int_A P U_{bn} dA = P_3\, A\, \frac{dh}{dt} \tag{7.II.h}$$

The combination of all the terms for the tank results in

$$\frac{d}{dt}\left(\left[\frac{\overline{U_t}^2}{2} + \frac{g\,h}{2}\right]\overbrace{h\,A}^{V}\right) - \frac{1}{2}\left(\frac{dh}{dt}\right)^2\left(\frac{A_3}{A_1}\right)^2 U_1\ A_1 + \frac{K_t}{2\,\rho}\left(\frac{dh}{dt}\right)^2 = \frac{(P_3 - P_1)}{\rho} \tag{7.II.i}$$

**Pipe Control Volume**

The analysis of the liquid in the pipe is similar to Example 7.1. The conservation of the liquid in the pipe is the same as in Example 7.1 and thus equation (7.I.a) is used

$$U_1 = U_2 \tag{7.II.j}$$

$$U_p + \frac{4\,\rho\, L\, \pi\, r^2}{K_p}\frac{d\,U_p}{dt} = \frac{2\,(P_1 - P_2)}{K_p} \tag{7.II.k}$$

where $K_p$ is the resistance in the pipe and $U_p$ is the (averaged) velocity in the pipe. Using equation (7.II.c) eliminates the $U_p$ as

$$\frac{dh}{dt} + \frac{4\,\rho\, L\, \pi\, r^2}{K}\frac{d^2\,h}{dt^2} = \left(\frac{R}{r}\right)^2 \frac{2\,(P_1 - P_2)}{K_p} \tag{7.II.l}$$

Equation (7.II.l) can be rearranged as

$$\frac{K_p}{2\,\rho}\left(\frac{r}{R}\right)^2\left(\frac{dh}{dt} + \frac{4\,\rho\, L\, \pi\, r^2}{K}\frac{d^2\,h}{dt^2}\right) = \frac{(P_1 - P_2)}{\rho} \tag{7.II.m}$$

**Solution**

The equations (7.II.m) and (7.II.i) provide the frame in which the liquid velocity in tank and pipe have to be solved. In fact, it can be noticed that the liquid velocity in the tank is related to the height and the liquid velocity in the pipe. Thus, there is only one equation with one unknown. The relationship between the height was obtained by substituting equation (7.II.c) in equation (7.II.m). The equations (7.II.m) and (7.II.i) have two unknowns ($dh/dt$ and $P_1$) which are sufficient to solve the problem. It can be noticed that two initial conditions are required to solve the problem.

The governing equation obtained by from adding equation (7.II.m) and (7.II.i) as

$$\begin{aligned}\frac{d}{dt}\left(\left[\frac{\overline{U_t}^2}{2} + \frac{g\,h}{2}\right]\right.&\left.\overbrace{h\,A}^{V}\right) - \frac{1}{2}\left(\frac{dh}{dt}\right)^2\left(\frac{A_3}{A_1}\right)^2 U_1\ A_1 + \frac{K_t}{2\,\rho}\left(\frac{dh}{dt}\right)^2 \\ &+\frac{K_p}{2\,\rho}\left(\frac{r}{R}\right)^2\left(\frac{dh}{dt} + \frac{4\,\rho\, L\, \pi\, r^2}{K}\frac{d^2\,h}{dt^2}\right) = \frac{(P_3 - P_2)}{\rho}\end{aligned} \tag{7.II.n}$$

The initial conditions are that zero initial velocity in the tank and pipe. Additionally, the height of liquid is at prescript point as

$$\begin{aligned} h(0) = & \quad h_0 \\ \frac{dh}{dt}(0) = & \quad 0 \end{aligned} \tag{7.II.o}$$

The solution of equation can be obtained using several different numerical techniques. The dimensional analysis method can be used to obtain solution various situations which will be presented later on.

End Solution

## Qualitative Questions

- A liquid flows in and out from a long pipe with uniform cross section as single phase. Assume that the liquid is slightly compressible. That is the liquid has a constant bulk modulus, $B_T$. What is the direction of the heat from the pipe or in to the pipe. Explain why the direction based on physical reasoning. What kind of internal work the liquid performed. Would happen when the liquid velocity is very large? What it will be still correct.

- A different liquid flows in the same pipe. If the liquid is compressible what is the direction of the heat to keep the flow isothermal?

- A tank is full of incompressible liquid. A certain point the tank is punctured and the liquid flows out. To keep the tank at uniform temperature what is the direction of the heat (from the tank or to the tank)?

# Part II

# Differential Analysis

# CHAPTER 8

# Differential Analysis

## 8.1 Introduction

The integral analysis has limited accuracy, which leads to a different approach of differential analysis. The differential analysis allows the investigation of the flow field in greater detail. In differential analysis, the emphasis is on infinitesimal scale and thus the analysis provides better accuracy[1]. This analysis leads to partial differential equations which are referred to as the Navier-Stokes equations. These equations are named after Claude–Louis Navier–Marie and George Gabriel Stokes. Like many equations they were independently derived by several people. First these equations were derived by Claude–Louis–Marie Navier as it is known in 1827. As usual Simon-Denis Poisson independently, as he done to many other equations or conditions, derived these equations in 1831 for the same arguments as Navier. The foundations for their arguments or motivations are based on a molecular view of how stresses are exerted between fluid layers. Barré de Saint Venant (1843) and George Gabriel Stokes (1845) derived these equation based on the relationship between stress and rate–of–strain (this approach is presented in this book).

Navier-Stokes equations are non–linear and there are more than one possible solution in many cases (if not most cases) e.g. the solution is not unique. A discussion about the "regular" solution is present and a brief discussion about limitations when the solution is applicable. Later in the Chapters on Real Fluid and Turbulence, with a presentation of the "non–regular" solutions will be presented with the associated issues of stability. However even for the "regular" solution the mathematics is very complex. One of the approaches is to reduce the equations by eliminating the viscosity effects. The equations without the viscosity effects are referred to as the ideal flow equations (Euler Equations) which will be discussed in the next chapter. The concepts of Add

[1]Which can be view as complementary analysis to the integral analysis.

Mass and Add Force, which are easier to discuss when the viscosity is ignored, and will be presented in the Ideal Flow chapter. It has to be pointed out that the Add Mass and Add Force appear regardless to the viscosity. Historically, the complexity of the equations, on one hand, leads to approximations and consequently to the ideal flow approximation (equations) and on the other hand experimental solutions of Navier–Stokes equations. The connection between these two ideas or fields was done via introduction of the boundary layer theory by Prandtl which will be discussed as well.

Even for simple situations, there are cases when the complying with the boundary conditions leads to a discontinuity (shock or choked flow). These equations cannot satisfy the boundary conditions in other cases and in way the fluid pushes the boundary condition(s) further downstream (choked flow). These issues are discussed in Open Channel Flow and Compressible Flow chapters. Sometimes, the boundary conditions create instability which alters the boundary conditions itself which is known as Interfacial instability. The choked flow is associated with a single phase flow (even the double choked flow) while the Interfacial instability associated with the Multi–Phase flow. This phenomenon is presented in Multi–phase chapter and in this chapter.

## 8.2 Mass Conservation

Fluid flows into and from a three dimensional infinitesimal control volume depicted in Figure 8.1. At a specific time this control volume can be viewed as a system. The mass conservation for this infinitesimal small system is zero thus

$$\frac{D}{Dt}\int_V \rho dV = 0 \tag{8.1}$$

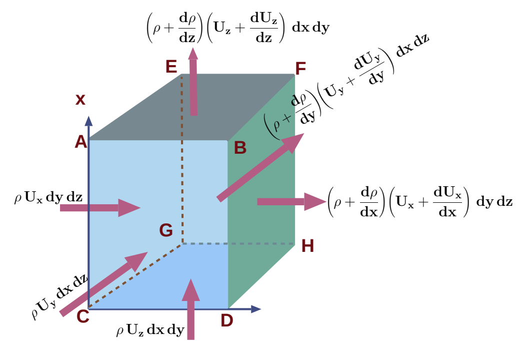



*Fig. -8.1. The mass balance on the infinitesimal control volume.*

However for a control volume using Reynolds Transport Theorem (RTT), the following can be written

$$\frac{D}{Dt}\int_V \rho dV = \frac{d}{dt}\int_V \rho dV + \int_A U_{rn}\,\rho\,dA = 0 \tag{8.2}$$

For a constant control volume, the derivative can enter into the integral (see also for the divergence theorem in the appendix A.1.2) on the right hand side and hence

$$\overbrace{\int_V \frac{d\rho}{dt}\,dV}^{\frac{d\rho}{dt}\,dV} + \int_A U_{rn}\,\rho\,dA = 0 \tag{8.3}$$

The first term in equation (8.3) for the infinitesimal volume is expressed, neglecting higher order derivatives, as

$$\int_V \frac{d\rho}{dt}\, dV = \frac{d\rho}{dt}\, \overbrace{dx\, dy\, dz}^{dV} + \overbrace{f\left(\frac{d^2\rho}{dt^2}\right) + \cdots}^{\sim 0} \tag{8.4}$$

The second term in the LHS of equation (8.2) is expressed[2] as

$$\begin{aligned}\int_A U_{rn}\,\rho\, dA = \; & \overbrace{dy\, dz}^{dA_{yz}} \left[ \left.(\rho\, U_x)\right|_x - \left.(\rho\, U_x)\right|_{x+dx} \right] + \\ & \overbrace{dx\, dz}^{dA_{xz}} \left[ \left.(\rho\, U_y)\right|_y - \left.(\rho\, U_y)\right|_{y+dy} \right] + \overbrace{dx\, dy}^{dA_{xz}} \left[ \left.(\rho\, U_z)\right|_z - \left.(\rho\, U_z)\right|_{z+dz} \right]\end{aligned} \tag{8.5}$$

The difference between point $x$ and $x+dx$ can be obtained by developing Taylor series as

$$\left.(\rho\, U_x)\right|_{x+dx} = \left.(\rho\, U_x)\right|_x + \left.\frac{\partial\,(\rho\, U_x)}{\partial x}\right|_x dx \tag{8.6}$$

The same can be said for the $y$ and $z$ coordinates. It also can be noticed that, for example, the operation, in the $x$ coordinate, produces additional $dx$ thus a infinitesimal volume element $dV$ is obtained for all directions. The combination can be divided by $dx\, dy\, dz$ and simplified by using the definition of the partial derivative in the regular process to be

$$\int_A U_{rn}\,\rho\, dA = -\left[\frac{\partial(\rho\, U_x)}{\partial x} + \frac{\partial(\rho\, U_y)}{\partial y} + \frac{\partial(\rho\, U_z)}{\partial z}\right] \tag{8.7}$$

Combining the first term with the second term results in the continuity equation in Cartesian coordinates as

Continuity in Cartesian Coordinates

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho\, U_x}{\partial x} + \frac{\partial\rho\, U_y}{\partial y} + \frac{\partial\rho\, U_z}{\partial z} = 0 \tag{8.8}$$

**Cylindrical Coordinates**

The same equation can be derived in cylindrical coordinates. The net mass change, as depicted in Figure 8.2, in the control volume is

$$d\,\dot{m} = \frac{\partial\rho}{\partial t}\overbrace{dr\, dz\, r\, d\theta}^{dv} \tag{8.9}$$

[2] Note that some time the notation $dA_{yz}$ also refers to $dA_x$.

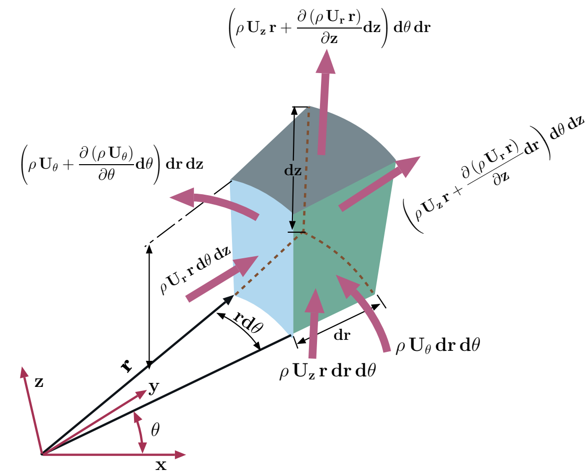



*Fig. -8.2. The mass conservation in cylindrical coordinates.*

The net mass flow out or in the $\widehat{\mathbf{r}}$ direction has an additional term which is the area change compared to the Cartesian coordinates. This change creates a different differential equation with additional complications. The change is

$$\begin{pmatrix}\text{flux in } r \\ \text{direction}\end{pmatrix} = d\theta\,dz\left(r\,\rho\,U_r - \left(r\,\rho\,U_r + \frac{\partial\rho\,U_r\,r}{\partial r}dr\right)\right) \tag{8.10}$$

The net flux in the $r$ direction is then

$$\begin{pmatrix}\text{net flux in the} \\ r \text{ direction}\end{pmatrix} = d\theta\,dz\,\frac{\partial\rho\,U_r\,r}{\partial r}dr \tag{8.11}$$

Note[3] that the $r$ is still inside the derivative since it is a function of $r$, e.g. the change of $r$ with $r$. In a similar fashion, the net flux in the $z$ coordinate be written as

$$\text{net flux in } z \text{ direction} = r\,d\theta\,dr\,\frac{\partial\left(\rho\,U_z\right)}{\partial z}dz \tag{8.12}$$

The net change in the $\theta$ direction is then

$$\text{net flux in } \theta \text{ direction} = dr\,dz\,\frac{\partial\rho\,U_\theta}{\partial\theta}d\theta \tag{8.13}$$

Combining equations (8.11)–(8.13) and dividing by infinitesimal control volume, $dr\;r\,d\theta\,dz$, results in

$$\begin{pmatrix}\text{total} \\ \text{net flux}\end{pmatrix} = -\left(\frac{1}{r}\frac{\partial\left(\rho\,U_r\,r\right)}{\partial r} + \frac{\partial\rho\,U_z\,r}{\partial z} + \frac{\partial\rho\,U_\theta}{\partial\theta}\right) \tag{8.14}$$

[3]The mass flow is $\rho\,U_r\,r\,d\theta\,dz$ at $r$ point. Expansion to Taylor serious $\rho\,U_r\,r\,d\theta\,dz|_{r+dr}$ is obtained by the regular procedure. The mass flow at $r+dr$ is $\rho\,U_r\,r\,d\theta\,dz|_r + d/dr\left(\rho\,U_r\,r\,d\theta\,dz\right)dr + \cdots$. Hence, the $r$ is "trapped" in the derivative.

Combining equation (8.14) with the change in the control volume (8.9) divided by infinitesimal control volume, $dr\, r\, d\theta\, dz$ yields

Continuity in Cylindrical Coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial\left(r\,\rho\,U_r\right)}{\partial r} + \frac{1}{r}\frac{\partial\rho\,U_\theta}{\partial\theta} + \frac{\partial\rho\,U_z}{\partial z} = 0 \tag{8.15}$$

Carrying similar operations for the spherical coordinates, the continuity equation becomes

Continuity in Spherical Coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial\left(r^2\,\rho\,U_r\right)}{\partial r} + \frac{1}{r\,\sin\theta}\frac{\partial\left(\rho\,U_\theta\,\sin\theta\right)}{\partial\theta} + \frac{1}{r\,\sin\theta}\frac{\partial\rho\,U_\phi}{\partial z} = 0 \tag{8.16}$$

The continuity equations (8.8), (8.15) and (8.16) can be expressed in different coordinates. It can be noticed that the second part of these equations is the divergence (see the Appendix A.1.2 page 310). Hence, the continuity equation can be written in a general vector form as

Continuity Equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho\,\boldsymbol{U}\right) = 0 \tag{8.17}$$

*Advance material can be skipped*

The mass equation can be written in index notation for Cartesian coordinates. The index notation really does not add much to the scientific understanding. However, this writing reduce the amount of writing and potentially can help think about the problem or situation in more conceptional way. The mass equation (see in the appendix for more information on the index notation) written as

$$\frac{\partial\rho}{\partial t} + \frac{\partial\left(\rho\,U\right)_i}{\partial x_i} = 0 \tag{8.18}$$

Where $i$ is is of the $i$, $j$, and $k$[4]. Compare to equation (8.8). Again remember that the meaning of repeated index is summation.

*End Advance material*

The use of these equations is normally combined with other equations (momentum and or energy equations). There are very few cases where this equation is used on its own merit. For academic purposes, several examples are constructed here.

[4]notice the irony the second $i$ is the dirction and first $i$ is for any one of direction x(i), y(j), and z(k).

### 8.2.1 Mass Conservation Examples

Example 8.1:

*A layer of liquid has an initial height of $H_0$ with an uniform temperature of $T_0$. At time, $t_0$, the upper surface is exposed to temperature $T_1$ (see Figure 8.3). Assume that the actual temperature is exponentially approaches to a linear temperature profile as depicted in Figure 8.3. The density is a function of the temperature according to*

$$\frac{T - T_0}{T_1 - T_0} = \alpha \left(\frac{\rho - \rho_0}{\rho_1 - \rho_0}\right) \tag{8.I.a}$$

*where $\rho_1$ is the density at the surface and where $\rho_0$ is the density at the bottom. Assume that the velocity is only a function of the $y$ coordinate. Calculates the velocity of the liquid. Assume that the velocity at the lower boundary is zero at all times. Neglect the mutual dependency of the temperature and the height.*

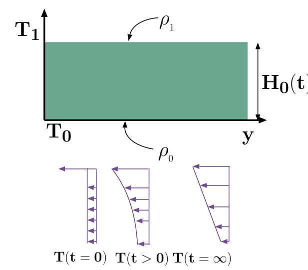



*Fig. -8.3. Mass flow due to temperature difference for example 8.1*

<u>Solution</u>

The situation is unsteady state thus the unsteady state and one dimensional continuity equation has to be used which is

$$\frac{\partial \rho}{\partial t} + \frac{\partial\left(\rho U_y\right)}{\partial y} = 0 \tag{8.I.b}$$

with the boundary condition of zero velocity at the lower surface $U_y(y=0)=0$. The expression that connects the temperature with the space for the final temperature as

$$\frac{T - T_0}{T_1 - T_0} = \alpha \, \frac{H_0 - y}{H_0} \tag{8.I.c}$$

The exponential decay is $\left(1 - e^{-\beta\, t}\right)$ and thus the combination (with equation (8.I.a)) is

$$\frac{\rho - \rho_0}{\rho_1 - \rho_0} = \alpha \, \frac{H_0 - y}{H_0} \left(1 - e^{-\beta\, t}\right) \tag{8.I.d}$$

Equation (8.I.d) relates the temperature with the time and the location was given in the question (it is not the solution of any model). It can be noticed that the height $H_0$ is a function of time. For this question, it is treated as a constant. Substituting the density, $\rho$, as a function of time into the governing equation (8.I.b) results in

$$\overbrace{\alpha\, \beta \left(\frac{H_0 - y}{H_0}\right) e^{-\beta\, t}}^{\frac{\partial \rho}{\partial t}} + \overbrace{\frac{\partial \left(U_y\, \alpha\, \frac{H_0 - y}{H_0} \left(1 - e^{-\beta\, t}\right)\right)}{\partial y}}^{\frac{\partial \rho\, U_y}{\partial y}} = 0 \tag{8.I.e}$$

Equation (8.I.e) is first order ODE with the boundary condition $U_y(y=0)=0$ which can be arranged as

$$\frac{\partial\left(U_y\,\alpha\,\frac{H_0-y}{H_0}\left(1-e^{-\beta\,t}\right)\right)}{\partial y}=-\alpha\,\beta\left(\frac{H_0-y}{H_0}\right)e^{-\beta\,t} \tag{8.I.f}$$

$U_y$ is a function of the time but not $y$. Equation (8.I.f) holds for any time and thus, it can be treated for the solution of equation (8.I.f) as a constant[5]. Hence, the integration with respect to $y$ yields

$$\left(U_y\,\alpha\,\frac{H_0-y}{H_0}\left(1-e^{-\beta\,t}\right)\right)=-\alpha\,\beta\left(\frac{2\,H_0-y}{2\,H_0}\right)e^{-\beta\,t}y+c \tag{8.I.g}$$

Utilizing the boundary condition $U_y(y=0)=0$ yields

$$\left(U_y\,\alpha\,\frac{H_0-y}{H_0}\left(1-e^{-\beta\,t}\right)\right)=-\alpha\,\beta\left(\frac{2\,H_0-y}{2\,H_0}\right)e^{-\beta\,t}\left(y-1\right) \tag{8.I.h}$$

or the velocity is

$$U_y=\beta\left(\frac{2\,H_0-y}{2\,\left(H_0-y\right)}\right)\frac{e^{-\beta\,t}}{\left(1-e^{-\beta\,t}\right)}\left(1-y\right) \tag{8.I.i}$$

It can be noticed that indeed the velocity is a function of the time and space $y$.

End Solution

### 8.2.2 Simplified Continuity Equation

A simplified equation can be obtained for a steady state in which the transient term is eliminated as (in vector form)

$$\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{U}\right)=0 \tag{8.19}$$

If the fluid is incompressible then the governing equation is a volume conservation as

$$\boldsymbol{\nabla}\cdot\boldsymbol{U}=0 \tag{8.20}$$

Note that this equation appropriate only for a single phase case.

Example 8.2:

*In many coating processes a thin film is created by a continuous process in which liquid injected into a moving belt that carries the material out as exhibited in Figure 8.4.*

[5]Since the time can be treated as a constant for $y$ integration.

*The temperature and mass transfer taking place which reduces (or increases) the thickness of the film. For this example, assume that no mass transfer occurs or can be neglected and the main mechanism is heat transfer. Assume that the film temperature is only a function of the distance from the extraction point. Calculate the film velocity field if the density is a function of the temperature. The relationship between the density and the temperature is linear as*

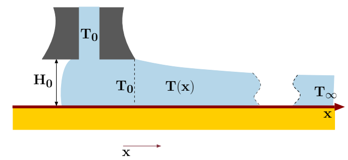



*Fig. -8.4. Mass flow in coating process for example 8.2.*

$$\frac{\rho - \rho_\infty}{\rho_0 - \rho_\infty} = \alpha \left( \frac{T - T_\infty}{T_0 - T_\infty} \right) \tag{8.II.a}$$

*State your assumptions.*

Solution

This problem is somewhat similar to Example 8.1[6], however it can be considered as steady state. At any point the governing equation in coordinate system that moving with the belt is

$$\frac{\partial \left(\rho \, U_x\right)}{\partial x} + \frac{\partial \left(\rho \, U_y\right)}{\partial y} = 0 \tag{8.II.b}$$

At first, it can be assumed that the material moves with at the belt in the $x$ direction in the same velocity. This assumption is consistent with the first solution (no stability issues). If the frame of reference was moving with the belt then there is only velocity component in the $y$ direction[7]. Hence equation (8.II.b) can be written as

$$U_x \frac{\partial \rho}{\partial x} = -\frac{\partial \left(\rho \, U_y\right)}{\partial y} \tag{8.II.c}$$

Where $U_x$ is the belt velocity.

See the resembles to equation (8.I.b). The solution is similar to the previous Example 8.1 for the general function $T = F(x)$.

$$\frac{\partial \rho}{\partial x} = \frac{\alpha}{U_x} \frac{\partial F(x)}{\partial x} \left(\rho_0 - \rho_\infty\right) \tag{8.II.d}$$

Substituting this relationship in equation (8.II.d) into the governing equation results in

$$\frac{\partial U_y \, \rho}{\partial y} = \frac{\alpha}{U_x} \frac{\partial F(x)}{\partial x} \left(\rho_0 - \rho_\infty\right) \tag{8.II.e}$$

[6]The presentation of one dimension time dependent problem to two dimensions problems can be traced to heat and mass transfer problems. One of the early pioneers who suggest this idea is Higbie which Higbie's equation named after him. Higbie's idea which was rejected by the scientific establishment. He spend the rest of his life to proof it and ending in a suicide. On personal note, this author Master thesis is extension Higbie's equation.

[7]In reality this assumption is correct only in a certain range. However, the discussion about this point is beyond the scope of this section.

The density is expressed by equation (8.II.a) and thus

$$U_y = \frac{\alpha}{\rho\, U_x} \frac{\partial F(x)}{\partial x} \left(\rho_0 - \rho_\infty\right) y + c \tag{8.II.f}$$

Notice that $\rho$ could "come" out of the derivative (why?) and move into the RHS. Applying the boundary condition $U_y(t = 0) = 0$ results in

$$U_y = \frac{\alpha}{\rho(x)\, U_x} \frac{\partial F(x)}{\partial x} \left(\rho_0 - \rho_\infty\right) y \tag{8.II.g}$$

End Solution

Example 8.3:

*The velocity in a two dimensional field is assumed to be in a steady state. Assume that the density is constant and calculate the vertical velocity ($y$ component) for the following $x$ velocity component.*

$$U_x = a\, x^2 + b\, y^2 \tag{8.III.a}$$

*Next, assume the density is also a function of the location as*

$$\rho = m\, e^{x+y} \tag{8.III.b}$$

*Where $m$ is constant. Calculate the velocity field in this case.*

Solution

The flow field must comply with the mass conservation (8.20) thus

$$2\, a\, x + \frac{\partial U_y}{\partial y} = 0 \tag{8.III.c}$$

Equation (8.III.c) is an ODE with constant coefficients. It can be noted that $x$ should be treated as a constant parameter for the $y$ coordinate. Thus,

$$U_y = -\int 2\, a\, x + f(x) = -2\, x\, y + f(x) \tag{8.III.d}$$

The integration constant in this case is not really a constant but rather an arbitrary function of $x$. Notice the symmetry of the situation. The velocity, $U_x$ has also arbitrary function in the $y$ component.

For the second part equation (8.19) is applicable and used as

$$\frac{\partial \left(a\, x^2 + b\, y^2\right) \left(m\, e^{x+y}\right)}{\partial x} + \frac{\partial\, U_y\, \left(m\, e^{x+y}\right)}{\partial y} = 0 \tag{8.III.e}$$

Taking the derivative of the first term and second part move the other side results in

$$a\left(2\,x+x^2+\frac{b}{a}\,y^2\right)\,e^{x+y}=-\left(e^{x+y}\right)\,\left(\frac{\partial\,U_y}{\partial y}+U_y\right) \tag{8.III.f}$$

The exponent can be canceled to simplify further the equation (8.III.f) and switching sides to be

$$\left(\frac{\partial\,U_y}{\partial y}+U_y\right)=-a\,\left(2\,x+x^2+\frac{b}{a}\,y^2\right) \tag{8.III.g}$$

Equation (8.III.g) is first order ODE that can be solved by combination of the homogeneous solution with the private solution (see for explanation in the Appendix). The homogeneous equation is

$$\left(\frac{\partial\,U_y}{\partial y}+U_y\right)=0 \tag{8.III.h}$$

The solution for (8.III.h) is $U_y = c\,e^{-y}$ (see for explanation in the appendix). The private solution is

$$U_y|_{private}=\left(-b\,\left(y^2-2\,y+2\right)-a\,x^2-2\,a\,x\right) \tag{8.III.i}$$

The total solution is

$$U_y=c\,e^{-y}+\left(-b\,\left(y^2-2\,y+2\right)-a\,x^2-2\,a\,x\right) \tag{8.III.j}$$

End Solution

Example 8.4:
*Can the following velocities co-exist.*

$$U_x=(x\,t)^2\;z \qquad U_y=(x\,t)+(y\,t)+(z\,t) \qquad U_z=(x\,t)+(y\,t)+(z\,t) \tag{8.IV.a}$$

*Is the flow is incompressible? Is the flow in a steady state condition?*

SOLUTION

Whether the solution is in a steady state or not can be observed from whether the velocity contains time component. Thus, this flow field is not steady state it contains time componnet. This continuity equation is checked if the flow incompressible (constant density). The derivative of each componnet are

$$\frac{\partial U_x}{\partial x}=t^2\,z \qquad \frac{\partial U_y}{\partial y}=t \qquad \frac{\partial U_z}{\partial z}=t \tag{8.IV.b}$$

Or gradient or the combination of these derivatives is

$$\nabla\boldsymbol{U}=t^2\,z+2\,t \tag{8.IV.c}$$

The divergence isn't zero thus this flow, if it exist, must be compressible flow. This flow can exist only for a limit time since over time the divergence is unbounded (infinite source).

End Solution

Example 8.5:
*Find the density as a function of the time for a given one dimensional flow of* $U_x = x\,e^{5\,\alpha\,y}\,(\cos(\alpha\,t))$. *The initial density is* $\rho(t=0) = \rho_0$.

SOLUTION

This problem is one dimensional unsteady state and for a compressible substance. Hence, the mass conservation is reduced only for one dimensional form as

$$\frac{\partial\rho}{\partial t} + \frac{\partial\left(U_x\,\rho\right)}{\partial x} = 0 \tag{8.V.a}$$

Mathematically speaking, this kind of presentation is possible. However physically there are velocity components in $y$ and $z$ directions. In this problem, these physical components are ignored for academic reasons. Equation (8.V.a) is first order partial differential equation which can be converted to an ordinary differential equations when the velocity component, $U_x$, is substituted. Using,

$$\frac{\partial U_x}{\partial x} = e^{5\,\alpha\,y}\,\left(\cos\left(\alpha\,t\right)\right) \tag{8.V.b}$$

Substituting equation (8.V.b) into equation (8.V.a) and noticing that the density, $\rho$, is a function of $x$ results of

$$\frac{\partial\rho}{\partial t} = -\rho\,x\,e^{5\,\alpha\,y}\,\left(\cos\left(\alpha\,t\right)\right) - \frac{\partial\rho}{\partial x}\,e^{5\,\alpha\,y}\,\left(\cos\left(\alpha\,t\right)\right) \tag{8.V.c}$$

Equation (8.V.c) can be separated to yield

$$\overbrace{\frac{1}{\cos\left(\alpha\,t\right)}\,\frac{\partial\rho}{\partial t}}^{f(t)} = \overbrace{-\rho\,x\,e^{5\,\alpha\,y} - \frac{\partial\rho}{\partial x}\,e^{5\,\alpha\,y}}^{f(y)} \tag{8.V.d}$$

A possible solution is when the left and the right hand sides are equal to a constant. In that case the left hand side is

$$\frac{1}{\cos\left(\alpha\,t\right)}\,\frac{\partial\rho}{\partial t} = c_1 \tag{8.V.e}$$

The solution of equation (8.V.e) is reduced to ODE and its solution is

$$\rho = \frac{c_1\,sin\left(\alpha\,t\right)}{\alpha} + c_2 \tag{8.V.f}$$

The same can be done for the right hand side as

$$\rho\, x\, e^{5\,\alpha\, y} + \frac{\partial \rho}{\partial x}\, e^{5\,\alpha\, y} = c_1 \tag{8.V.g}$$

The term $e^{5\,\alpha\, y}$ is always positive, real value, and independent of $y$ thus equation (8.V.g) becomes

$$\rho\, x + \frac{\partial \rho}{\partial x} = \frac{c_1}{e^{5\,\alpha\, y}} = c_3 \tag{8.V.h}$$

Equation (8.V.h) is a constant coefficients first order ODE which its solution discussed extensively in the appendix. The solution of (8.V.h) is given by

$$\rho = e^{-\frac{x^2}{2}} \left( c - \overbrace{\frac{\sqrt{\pi}\, i\, c_3\, erf\left(\frac{i\,x}{\sqrt{2}}\right)}{\sqrt{2}}}^{\text{impossible solution}} \right) \tag{8.V.i}$$

which indicates that the solution is a complex number thus the constant, $c_3$, must be zero and thus the constant, $c_1$ vanishes as well and the solution contain only the homogeneous part and the private solution is dropped

$$\rho = c_2\, e^{-\frac{x^2}{2}} \tag{8.V.j}$$

The solution is the multiplication of equation (8.V.j) by (8.V.f) transfered to

$$\rho = c_2\, e^{-\frac{x^2}{2}} \left( \frac{c_1\, sin\,(\alpha\, t)}{\alpha} + c_2 \right) \tag{8.V.k}$$

Where the constant, $c_2$, is an arbitrary function of the $y$ coordinate.

End Solution

## 8.3 Conservation of General Quantity

### 8.3.1 Generalization of Mathematical Approach for Derivations

In this section a general approach for the derivations for conservation of any quantity e.g. scalar, vector or tensor, are presented. Suppose that the property $\phi$ is under a study which is a function of the time and location as $\phi(x, y, z, t)$. The total amount of quantity that exist in arbitrary system is

$$\Phi = \int_{sys} \phi\, \rho\, dV \tag{8.21}$$

Where $\Phi$ is the total quantity of the system which has a volume $V$ and a surface area of $A$ which is a function of time. A change with time is

$$\frac{D\Phi}{Dt} = \frac{D}{Dt} \int_{sys} \phi\, \rho\, dV \tag{8.22}$$

Using RTT to change the system to a control volume (see equation (5.33)) yields

$$\frac{D}{Dt}\int_{sys}\phi\,\rho\,dV = \frac{d}{dt}\int_{cv}\phi\,\rho\,dV + \int_{A}\rho\,\phi\boldsymbol{U}\cdot dA \tag{8.23}$$

The last term on the RHS can be converted using the divergence theorem (see the appendix[8]) from a surface integral into a volume integral (alternatively, the volume integral can be changed to the surface integral) as

$$\int_{A}\rho\,\phi\boldsymbol{U}\cdot dA = \int_{V}\nabla\cdot\left(\rho\,\phi\boldsymbol{U}\right)dV \tag{8.24}$$

Substituting equation (8.24) into equation (8.23) yields

$$\frac{D}{Dt}\int_{sys}\phi\,\rho\,dV = \frac{d}{dt}\int_{cv}\phi\,\rho\,dV + \int_{cv}\nabla\cdot\left(\rho\,\phi\boldsymbol{U}\right)dV \tag{8.25}$$

Since the volume of the control volume remains independent of the time, the derivative can enter into the integral and thus combining the two integrals on the RHS results in

$$\frac{D}{Dt}\int_{sys}\phi\,\rho\,dV = \int_{cv}\left(\frac{d\left(\phi\,\rho\right)}{dt} + \nabla\cdot\left(\rho\,\phi\boldsymbol{U}\right)\right)dV \tag{8.26}$$

The definition of equation (8.21) LHS can be changed to simply the derivative of $\Phi$. The integral is carried over arbitrary system. For an infinitesimal control volume the change is

$$\frac{D\,\Phi}{Dt} \cong \left(\frac{d\left(\phi\,\rho\right)}{dt} + \nabla\cdot\left(\rho\,\phi\boldsymbol{U}\right)\right)\overbrace{dx\,dy\,dz}^{dV} \tag{8.27}$$

## 8.3.2 Examples of Several Quantities

### 8.3.2.1 The General Mass Time Derivative

Using $\phi = 1$ is the same as dealing with the mass conservation. In that case $\frac{D\,\Phi}{Dt} = \frac{D\,\rho}{Dt}$ which is equal to zero as

$$\int\left(\frac{d\left(\overbrace{1}^{\phi}\;\rho\right)}{dt} + \nabla\cdot\left(\rho\,\overbrace{1}^{\phi}\,\boldsymbol{U}\right)\right)\overbrace{dx\,dy\,dz}^{dV} = 0 \tag{8.28}$$

[8]These integrals are related to RTT. Basically the divergence theorem relates the flow out (or) in and the sum of the all the changes inside the control volume.

The integral is over an arbitrary volume which means that integrand is zero as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = 0 \tag{8.29}$$

Equation (8.29) can be rearranged as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{U}\, \nabla \cdot \rho + \rho \nabla \cdot \boldsymbol{U} = 0 \tag{8.30}$$

Equation (8.30) can be further rearranged so derivative of the density is equal the divergence of velocity as

$$\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{U}\, \nabla \cdot \rho\right) = -\nabla \cdot \boldsymbol{U} \tag{8.31}$$

Equation (8.31) relates the density rate of change or the volumetric change to the velocity divergence of the flow field. The term in the bracket LHS is referred in the literature as substantial derivative. The substantial derivative represents the change rate of the density at a point which moves with the fluid.

**Acceleration Direct Derivations**

One of the important points is to find the fluid particles acceleration. A fluid particle velocity is a function of the location and time. Therefore, it can be written that

$$\boldsymbol{U}(x,y,z,t) = U_x(x,y,x,t)\,\widehat{i} + U_y(x,y,z,t)\,\widehat{j} + U_z(x,y,z,t)\,\widehat{k} \tag{8.32}$$

The acceleration will be

$$\frac{d\boldsymbol{U}}{dt} = \frac{d\,U_x}{dt}\,\widehat{i} + \frac{d\,U_y}{dt}\,\widehat{j} + \frac{d\,U_z}{dt}\,\widehat{k} \tag{8.33}$$

The velocity components are a function of four variables, ($x$, $y$, $z$, and $t$), and hence

$$\frac{d\,U_x}{dt} = \frac{\partial\,U_x}{\partial t}\overbrace{\frac{d\,t}{d\,t}}^{=1} + \frac{\partial\,U_x}{\partial x}\overbrace{\frac{d\,x}{d\,t}}^{U_x} + \frac{\partial\,U_x}{\partial y}\overbrace{\frac{d\,y}{d\,t}}^{U_y} + \frac{\partial\,U_x}{\partial z}\overbrace{\frac{d\,z}{d\,t}}^{U_z} \tag{8.34}$$

The acceleration in the $x$ can be written as

$$\frac{d\,U_x}{dt} = \frac{\partial\,U_x}{\partial t} + U_x\frac{\partial\,U_x}{\partial x} + U_y\frac{\partial\,U_x}{\partial y} + U_z\frac{\partial\,U_x}{\partial z} = \frac{\partial\,U_x}{\partial t} + (\boldsymbol{U}\cdot\nabla)\;U_x \tag{8.35}$$

The same can be developed to the other two coordinates which can be combined (in a vector form) as

$$\frac{d\boldsymbol{U}}{dt} = \frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U} \tag{8.36}$$

or in a more explicit form as

$$\frac{d\boldsymbol{U}}{dt} = \overbrace{\frac{\partial \boldsymbol{U}}{\partial t}}^{\text{local acceleration}} + \overbrace{\boldsymbol{U}\frac{\partial \boldsymbol{U}}{\partial x} + \boldsymbol{U}\frac{\partial \boldsymbol{U}}{\partial y} + \boldsymbol{U}\frac{\partial \boldsymbol{U}}{\partial z}}^{\text{convective acceleration}} \tag{8.37}$$

The time derivative referred in the literature as the local acceleration which vanishes when the flow is in a steady state. While the flow is in a steady state there is only convecive acceleration of the flow. The flow in a nozzle is an example to flow at steady state but yet has acceleration which flow with low velocity can achieve a supersonic flow.

## 8.4 Momentum Conservation

The relationship among the shear stress various components have to be established. The stress is a relationship between the force and area it is acting on or force divided by the area (division of vector by a vector). This division creates a tensor which the physical meaning will be explained here (the mathematical explanation can be found in the mathematical appendix of the book). The area has a direction or orientation which control the results of this division. So it can be written that

$$\boldsymbol{\tau} = f(\boldsymbol{F}, \boldsymbol{A}) \tag{8.38}$$

It was shown that in a static case (or in better words, when the shear stresses are absent) it was written

$$\boldsymbol{\tau} = -P\widehat{n} \tag{8.39}$$

It also was shown that the pressure has to be continuous. However, these stresses that act on every point and have three components on every surface and depend on the surface orientation. A common approach is to collect the stress in a "standard" orientation and then if needed the stresses can be reorientated to a new direction. The transformation is available because the "standard" surface can be transformed using trigonometrical functions. In Cartesian coordinates on surface in the $x$ direction the stresses are

$$\boldsymbol{\tau}^{(x)} = \quad \tau_{xx} \quad \tau_{xy} \quad \tau_{xz} \tag{8.40}$$

where $\tau_{xx}$ is the stress acting on surface $x$ in the $x$ direction, and $\tau_{xy}$ is the stress acting on surface $x$ in the $y$ direction, similarly for $\tau_{xz}$. The notation $\boldsymbol{\tau}^{(x_i)}$ is used to denote the stresses on $x_i$ surface. It can be noticed that no mathematical symbols are written between the components. The reason for this omission is that there is no physical meaning for it. Similar "vectors" exist for the $y$ and $z$ coordinates which can be written in a matrix form

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{8.41}$$

Suppose that a straight angle tetrahedron is under stress as shown in Figure 8.5. The forces balance in the $x$ direction excluding the slanted surface is

$$F_x = -\tau_{yx}\delta A_y - \tau_{xx}\delta A_x - \tau_{zx}\delta A_z \tag{8.42}$$

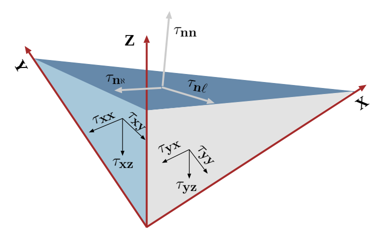



*Fig. -8.5. Stress diagram on a tetrahedron shape.*

where $\delta A_y$ is the surface area of the tetrahedron in the $y$ direction, $\delta A_x$ is the surface area of the tetrahedron in the $x$ direction and $\delta A_z$ is the surface area of the tetrahedron in the $z$ direction. The opposing forces which acting on the slanted surface in the $x$ direction are

$$F_x = \delta A_n \left( \tau_{nn}\, \widehat{n} \cdot \widehat{i} - \tau_{n\ell}\, \widehat{\ell} \cdot \widehat{i} - \tau_{n\aleph} \widehat{\aleph} \cdot \widehat{i} \right) \tag{8.43}$$

Where here $\widehat{\aleph}$, $\widehat{\ell}$ and $\widehat{n}$ are the local unit coordinates on $n$ surface the same can be written in the $x$, and $z$ directions. The transformation matrix is then

$$\begin{pmatrix} F_x \\ F_y \\ F_x \end{pmatrix} = \begin{pmatrix} \widehat{n} \cdot \widehat{i} & \widehat{\ell} \cdot \widehat{i} & \widehat{\aleph} \cdot \widehat{i} \\ \widehat{n} \cdot \widehat{j} & \widehat{\ell} \cdot \widehat{j} & \widehat{\aleph} \cdot \widehat{j} \\ \widehat{n} \cdot \widehat{k} & \widehat{\ell} \cdot \widehat{k} & \widehat{\aleph} \cdot \widehat{k} \end{pmatrix} \delta A_n \tag{8.44}$$

When the tetrahedron is shrunk to a point relationship of the stress on the two sides can be expended by Taylor series and keeping the first derivative. If the first derivative is neglected (tetrahedron is without acceleration) the two sides are related as

$$-\tau_{yx}\delta A_y - \tau_{xx}\delta A_x - \tau_{zx}\delta A_z = \delta A_n \left( \tau_{nn}\, \widehat{n} \cdot \widehat{i} - \tau_{n\ell}\, \widehat{\ell} \cdot \widehat{i} - \tau_{n\aleph} \widehat{\aleph} \cdot \widehat{i} \right) \tag{8.45}$$

The same can be done for $y$ and $z$ directions. The areas are related to each other through angles. These relationships provide the transformation for the different orientations which depends only angles of the orientations. This matrix is referred to as stress tensor and as it can be observed has nine terms.

### The Symmetry of the Stress Tensor

A small liquid cubical has three possible rotation axes. Here only one will be discussed the same conclustions can be drown on the other direction. The cubical rotation can involve two parts: one distortion and one rotation[9]. A finite angular

[9]For infinitesimal change the lines can be approximated as straight.

distortion of infinitesimal cube requires requires an infinite shear which required fore the infinite moment. Hence, the rotation of the infinitesimal fluid cube can be viewed as it is done almost as a solid body rotation. Balance of momentum around the $z$ direction shown in Figure 8.6 is

$$M_z = I_{zz} \frac{d\theta}{dt} \tag{8.46}$$

Where $M_z$ is the cubic moment around the cubic center and $I_{zz}$[10] is the moment of inertia around that center. The momentum can be assested by the shear stresses which act on it. The shear stress at point $x$ is $\tau_{xy}$. However, the shear stress at point $x + dx$ is

$$\tau_{xy}\big|_{x+dx} = \tau_{xy} + \frac{d\tau_{xy}}{dx} dx \tag{8.47}$$

The same can be said for $\tau_{yx}$ for $y$ direction. The clarity of this analysis can be improved if additional terms are taken, yet it turn out that the results will be the same. The normal body force (gravity) acts through the cubic center of gravity. The moment that creats by this action can be neglected (the changes are insignificant). However, for cases that body force, such as the magnetic fields, can create torque. For simplicity and generality, it is assumed that the external body force exerts a torque $G_T$ per unit volume at the specific location. The body force can exert torque is due to the fact that the body force is not uniform and hence not act through the mass center.

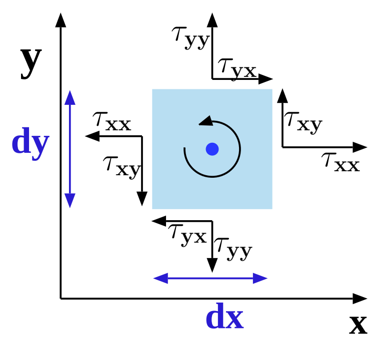



*Fig. -8.6. Diagram to analysis the shear stress tensor.*

*Advance material can be skipped*

The shear stress in the surface direction potentially can result in the torque due to the change in the shear stress[11]. For example, $\tau_{xx}$ at $x$ can be expended as a linear function

$$\tau_{xx} = \tau_{xx}\big|_y + \left.\frac{d\tau_{xx}}{dy}\right|_y \eta \tag{8.48}$$

where $\eta$ is the local coordinate in the $y$ direction stating at $y$ and "mostly used" between $y < \eta < y + dy$.

[10]See for the derivations in Example 3.5 for moment of inertia.

[11]This point bother this author in the completeness of the proof. It can be ignored, but provided to those who wonder why body forces can contribute to the torque while pressure, even though variyied, does not. This point is for self convincing since it deals with a "strange" and problematic "animals" of integral of infinitesimal length.

The moment that results from this shear force (clockwise positive) is

$$\int_y^{y+dy} \tau_{xx}(\eta) \left(\eta - \frac{dy}{2}\right) d\eta \tag{8.49}$$

Substituting (8.48) into (8.49) results

$$\int_y^{y+dy} \left( \tau_{xx}\big|_y + \left.\frac{d\tau_{xx}}{dy}\right|_y \eta \right) \left(\eta - \frac{dy}{2}\right) d\eta \tag{8.50}$$

The integral of (8.50) isn't zero (non symmetrical function around the center of integration). The reason that this term is neglected because on the other face of the cubic contributes an identical term but in the opposing direction (see Figure 8.6).

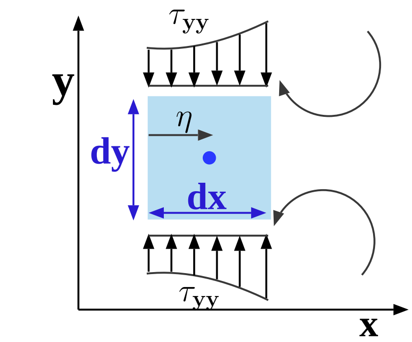



*Fig. -8.7. The shear stress creating torque.*

*End Advance material*

The net torque in the z-direction around the particle's center would then be

$$\left(\tau_{yx}\right) \frac{dx\,dy\,dz}{2} - \left(\tau_{yx} + \frac{\partial \tau_{xy}}{\partial x}\right) \frac{dx\,dy\,dz}{2} + \left(\tau_{xy}\right) \frac{dx\,dy\,dz}{2} - \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\right) \frac{dx\,dy\,dz}{2} = \overbrace{\rho\, dx\, dy\, dz \left((dx)^2 + (dy)^2\right)}^{I_{zz}} \frac{d\theta}{dt} \tag{8.51}$$

The actual components which contribute to the moment are

$$G_T + \tau_{xy} - \tau_{xy} + \overbrace{\frac{\partial\left(\tau_{yx} - \tau_{xy}\right)}{\partial y}}^{\cong 0} = \rho \frac{\underbrace{\left((dx)^2 + (dy)^2\right)}_{=0}}{12} \frac{d\theta}{dt} \tag{8.52}$$

which means since that $dx \longrightarrow 0$ and $dy \longrightarrow 0$ that

$$G_T + \tau_{xy} = \tau_{yx} \tag{8.53}$$

This analysis can be done on the other two directions and hence the general conclusion is that

$$G_T + \tau_{ij} = \tau_{ji} \tag{8.54}$$

where $i$ is one of $x, y, z$ and the $j$ is any of the other $x, y, z$[12]. For the case of $G_T = 0$ the stress tensor becomes symmetrical. The gravity is a body force that is considered in many kind of calculations and this force cause a change in symmetry of the stress

[12]The index notation is not the main mode of presentation in this book. However, since Potto Project books are used extensively and numerous people asked to include this notation it was added. It is believed that this notation should and can be used only after the physical meaning was "digested."

tensor. However, this change, for almost all practical purposes, can be neglected[13]. The magnetic body forces on the other hand is significant and has to be included in the calculations. If the body forces effect is neglected or do not exist in the problem then regardless the coordinate system orientation

$$\tau_{ij} = \tau_{ji} \qquad (i \neq j) \tag{8.55}$$

## 8.5 Derivations of the Momentum Equation

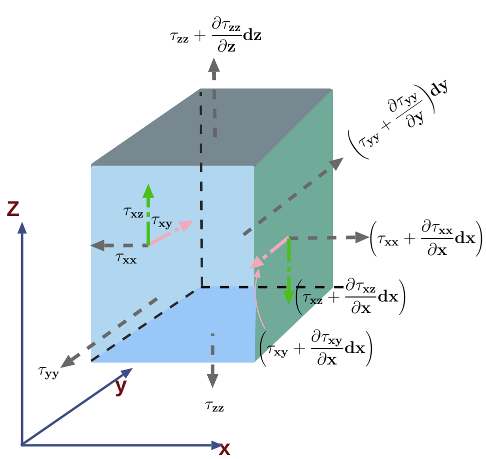



*Fig. -8.8. The shear stress at different surfaces. All shear stress shown in surface $x$ and $x+dx$.*

Previously it was shown that equation (6.11) is equivalent to Newton second law for fluids. Equation (6.11) is also applicable for the small infinitesimal cubic. One direction of the vector equation will be derived for $x$ Cartesian coordinate (see Figure 8.8). Later it will be used and generalized. For surface forces that acting on the cubic are surface forces, gravitation forces (body forces), and internal forces. The body force that acting on infinitesimal cubic in $x$ direction is

$$\widehat{i} \cdot \boldsymbol{f}_B = \boldsymbol{f}_{B\,x}\, dx\, dy\, dz \tag{8.56}$$

The dot product yields a force in the directing of $x$. The surface forces in $x$ direction on the $x$ surface on are

$$f_{xx} = \left.\tau_{xx}\right|_{x+dx} \times \overbrace{dy\, dz}^{dA_x} - \left.\tau_{xx}\right|_{x} \times \overbrace{dy\, dz}^{dA_x} \tag{8.57}$$

[13]In the Dimensional Analysis a discussion about this effect hopefully will be presented.

The surface forces in $x$ direction on the $y$ surface on are

$$f_{xy} = \tau_{yx}\big|_{y+dy} \times \overbrace{dx\,dz}^{dA_y} - \tau_{yx}\big|_{y} \times \overbrace{dx\,dz}^{dA_y} \tag{8.58}$$

The same can be written for the $z$ direction. The shear stresses can be expanded into Taylor series as

$$\tau_{ix}\big|_{i+di} = \tau_{ix} + \left.\frac{\partial\left(\tau_{ix}\right)}{\partial i}\right|_{i} di + \cdots \tag{8.59}$$

where $i$ in this case is $x$, $y$, or $z$. Hence, the total net surface force results from the shear stress in the $x$ direction is

$$f_x = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dx\,dy\,dz \tag{8.60}$$

after rearrangement equations such as (8.57) and (8.58) transformed into

$$\overbrace{\frac{DU_x}{Dt}\,\rho\,\cancel{dx}\,\cancel{dy}\,\cancel{dz}}^{\text{internal forces}} = \overbrace{\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\cancel{dx}\,\cancel{dy}\,\cancel{dz}}^{\text{surface forces}} + \overbrace{f_{G\,x}\,\rho\,\cancel{dx}\,\cancel{dy}\,\cancel{dz}}^{\text{body forces}} \tag{8.61}$$

equivalant equation (8.61) for $y$ coordinate is

$$\rho\,\frac{DU_y}{Dt} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho\, f_{G\,y} \tag{8.62}$$

The same can be obtained for the $z$ component

$$\rho\,\frac{DU_z}{Dt} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho\, f_{G\,z} \tag{8.63}$$

*Advance material can be skipped*

Generally the component momentum equation is as

$$\rho\,\frac{DU_i}{Dt} = \left(\frac{\partial \tau_{ii}}{\partial i} + \frac{\partial \tau_{ji}}{\partial j} + \frac{\partial \tau_{ki}}{\partial j}\right) + \rho\, f_{G\,i} \tag{8.64}$$

Where $i$ is the balance direction and $j$ and $k$ are two other coordinates. Equation (8.64) can be written in a vector form which combined all three components into one equation. The advantage of the vector from allows the usage of the different coordinates. The vector form is

$$\rho\,\frac{D\boldsymbol{U}}{Dt} = \nabla \cdot \boldsymbol{\tau}^{(i)} + \rho\,\boldsymbol{f_G} \tag{8.65}$$

where here

$$\boldsymbol{\tau}^{(i)} = \tau_{ix}\widehat{i} + \tau_{iy}\widehat{j} + \tau_{iz}\widehat{k}$$

is part of the shear stress tensor and $i$ can be any of the $x, y$, or $z$.

Or in index (Einstein) notation as

$$\rho \frac{DU_i}{Dt} = \frac{\partial \tau_{ji}}{\partial x_i} + \rho\, f_{G\,i} \tag{8.66}$$

*End Advance material*

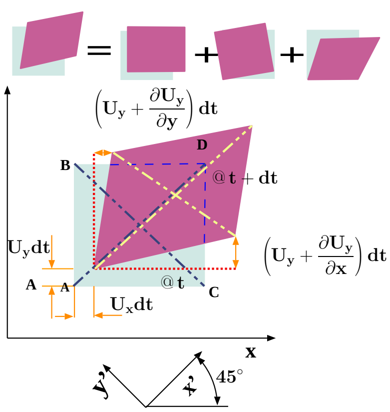



*Fig. -8.9. Control volume at $t$ and $t + dt$ under continuous angle deformation. Notice the three combinations of the deformation shown by purple color relative to blue color.*

Equations (8.61) or (8.62) or (8.63) requires that stress tensor be defined in term of the velocity/deformaiton. The relationship between the stress tensor and deformation depends on the classes of materials the stresses acts on. Additionally, the deformation can be viewed as a function of the velocity field. As engineers do in general, the simplest model is assumed which referred as the solid continuum model. In this model the (shear) stresses and rate of strains are assumed to be linearly related. In solid material, the shear stress yields a fix amount of deformation. In contrast, when applying the shear stress in fluids, the result is a continuous deformation. Furthermore, reduction of the shear stress does not return the material to its original state as in solids. The similarity to solids the increase shear stress in fluids yields larger deformations. Thus this "solid" model is a linear relationship with three main assumptions:

a. There is no preference in the orientation (also call isentropic fluid),

b. there is no left over stresses (In over words when the "no shear stress" situation exist the rate of deformation or strain is zero), and

c. a linear relationship between the shear stress to the rate of shear strain.

At time $t$, the control volume is at a square shape and at a location as depicted in Figure 8.9 (by the blue color). At time $t+dt$ the control volume undergoes three different changes. The control volume moves to a new location, rotates and changes the shape (the blow color in in Figure 8.9). The translational movement is referred to a movement of body without change of the body and without rotation. The rotation is the second movement that referred to a change in of the relative orientation inside the control volume. The third change is the misconfiguration or control volume (deformation). The deformation of the control volume has several components (see the top of Figure 8.9). The shear stress is related to the change in angle of the control volume lower left corner. The angle between $x$ to the new location of the control volume can be approximate for a small angle as

$$\frac{d\gamma_x}{dt} = \tan\left(\frac{U_y + \frac{dU_y}{dx}dx - U_y}{dx}\right) = \tan\left(\frac{dU_y}{dx}\right) \cong \frac{dU_y}{dx} \tag{8.67}$$

The total angle deformation (two sides $x$ and $y$) is

$$\frac{D\gamma_{xy}}{Dt} = \frac{dU_y}{dx} + \frac{dU_x}{dy} \tag{8.68}$$

In these derivatives, the symmetry $\frac{dU_y}{dx} \neq \frac{dU_x}{dy}$ was not assumed and or required because rotation of the control volume. However, under isentropic material it is assumed that the contribution of all the shear stresses contribute equally. For the assumption of a linear fluid[14].

$$\tau_{xy} = \mu\frac{D\gamma_{xy}}{Dt} = \mu\left(\frac{dU_y}{dx} + \frac{dU_x}{dy}\right) \tag{8.69}$$

where, $\mu$ is the "normal" or "ordinary" viscosity coefficient which the linear coefficient of proportionality and shear stress and it is assumed to be a property of the fluid. In a similar fashion it can be written to other directions for $x\,z$ as

$$\tau_{xz} = \mu\frac{D\gamma_{xz}}{Dt} = \mu\left(\frac{dU_z}{dx} + \frac{dU_x}{dz}\right) \tag{8.70}$$

and for the directions of $y\,z$ as

$$\tau_{yz} = \mu\frac{D\gamma_{yz}}{Dt} = \mu\left(\frac{dU_z}{dy} + \frac{dU_y}{dz}\right) \tag{8.71}$$

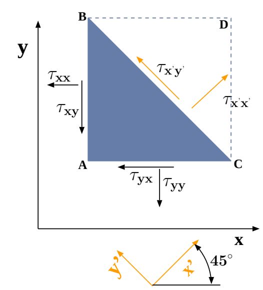



*Fig. -8.10. Shear stress at two coordinates in $45^\circ$ orientations.*

Note that the viscosity coefficient (the linear coefficient[15]) is assumed to be the same regardless of the direction. This assumption is referred as isotropic viscosity. It can be noticed at this stage, the relationship between the two of stress tensor was established. The only missing thing, at this stage, is the diagonal component which dealt below.

*Advance material can be skipped*

In general equation (8.69) can be written as

$$\tau_{ij} = \mu\frac{D\gamma_{ij}}{Dt} = \mu\left(\frac{dU_j}{di} + \frac{dU_i}{dj}\right) \tag{8.72}$$

[14]While not marked as important equation this equation is the source of the derivation.
[15]The first assumption was mentioned above.

where $i \neq j$ and $i = x$ or $y$ or $z$.

**Normal Stress**

The normal stress, $\tau_{ii}$ (where $i$ is either ,$x$, $y$, $z$) appears in shear matrix diagonal. To find the main (or the diagonal) stress the coordinates are rotate by $45°$. The diagonal lines (line *BC* and line *AD* in Figure 8.9) in the control volume move to the new locations. In addition, the sides *AB* and *AC* rotate in unequal amount which make one diagonal line longer and one diagonal line shorter. The normal shear stress relates to the change in the diagonal line length change. This relationship can be obtained by changing the coordinates orientation as depicted by Figure 8.10. The $dx$ is constructed so it equals to $dy$. The forces acting in the direction of $x$' on the ellement are combination of several terms. For example, on the "$x$" surface (lower surface) and the "$y$" (left) surface, the shear stresses are acting in this direction. It can be noticed that "$dx'$" surface is $\sqrt{2}$ times larger than $dx$ and $dy$ surfaces. The force balance in the $x'$ is

$$\overbrace{dy}^{A_x}\,\tau_{xx}\,\overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_x} + \overbrace{dx}^{A_y}\,\tau_{yy}\,\overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_y} + \overbrace{dx}^{A_y}\,\tau_{yx}\,\overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_y} + \overbrace{dy}^{A_x}\,\tau_{xy}\,\overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_y} = \overbrace{dx\sqrt{2}}^{A_{x'}}\,\tau_{x'x'} \tag{8.73}$$

dividing by $dx$ and after some rearrangements utilizing the identity $\tau_{xy} = \tau_{yx}$ results in

$$\frac{\tau_{xx} + \tau_{yy}}{2} + \tau_{yx} = \tau_{x'x'} \tag{8.74}$$

Setting the similar analysis in the $y'$ results in

$$\frac{\tau_{xx} + \tau_{yy}}{2} - \tau_{yx} = \tau_{y'y'} \tag{8.75}$$

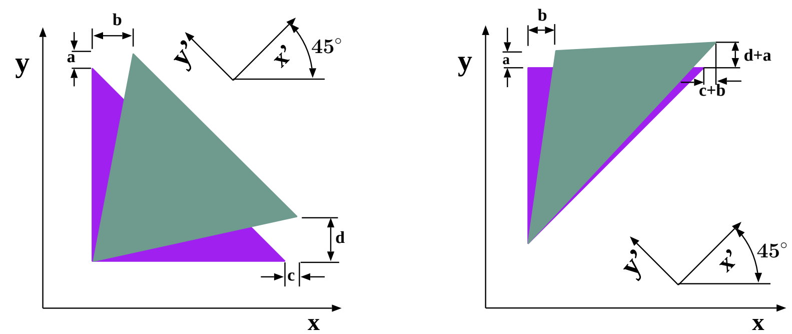
Subtracting (8.75) from (8.74) results in

$$2\,\tau_{yx} = \tau_{x'x'} - \tau_{y'y'} \tag{8.76}$$

or dividing by $2$ equation (8.76) becomes

$$\tau_{yx} = \frac{1}{2}\left(\tau_{x'x'} - \tau_{y'y'}\right) \tag{8.77}$$

Equation (8.76) relates the difference between the normal shear stress and the normal shear stresses in $x'$, $y'$ coordinates) and the angular strain rate in the regular ($x, y$ coordinates). The linear deformations in the $x'$ and $y'$ directions which is rotated $45°$ relative to the x and y axes can be expressed in both coordinates system. The angular strain rate in the ($x$, $y$) is frame related to the strain rates in the ($x'$, $y'$) frame. Figure 8.11(a) depicts the deformations of the triangular particles between time $t$ and




(a) Deformations of the isosceles triangular.

(b) Deformation of the straight angle triangle.

*Fig. -8.11. Different triangles deformation for the calculations of the normal stress.*

$t+dt$. The small deformations a , b, c, and d in the Figure are related to the incremental linear strains. The rate of strain in the $x$ direction is

$$d\epsilon_x = \frac{c}{dx} \tag{8.78}$$

The rate of the strain in $y$ direction is

$$d\epsilon_y = \frac{a}{dx} \tag{8.79}$$

The total change in the deformation angle is related to $\tan\theta$, in both sides $(d/dx+b/dy)$ which in turn is related to combination of the two sides angles. The linear angular deformation in $xy$ direction is

$$d\gamma_{xy} = \frac{b+d}{dx} \tag{8.80}$$

Here, $d\epsilon_x$ is the linear strain (increase in length divided by length) of the particle in the $x$ direction, and $d\epsilon_y$ is its linear strain in the y-direction. The linear strain in the $x'$ direction can be computed by observing Figure 8.11(b). The hypotenuse of the triangle is oriented in the x' direction (again observe Figure 8.11(b)). The original length of the hypotenuse $\sqrt{2}dx$. The change in the hypotenuse length is $\sqrt{(c+b)^2+(a+d)^2}$. It can be approximated that the change is about $45°$ because changes are infinitesimally small. Thus, $\cos 45°$ or $\sin 45°$ times the change contribute as first approximation to change. Hence, the ratio strain in the $x'$ direction is

$$d\epsilon_{x'} = \frac{\sqrt{(c+b)^2+(a+d)^2}}{\sqrt{2}dx} \simeq \frac{\dfrac{(c+b)}{\sqrt{2}} + \dfrac{(c+b)}{\sqrt{2}} + \overbrace{f\,(dx')}^{\sim 0}}{\sqrt{2}dx} \tag{8.81}$$

Equation (8.81) can be interpreted as (using equations (8.78), (8.79), and (8.80))

$$d\epsilon_{x'} = \frac{1}{2}\left(\frac{a+b+c+d}{dx}\right) = \frac{1}{2}\left(d\epsilon_y + d\epsilon_y + d\gamma_{xy}\right) \tag{8.82}$$

In the same fashion the strain, in $y'$ coordinate can be interpreted to be

$$d\epsilon_{y'} = \frac{1}{2}\left(d\epsilon_y + d\epsilon_y - d\gamma_{xy}\right) \tag{8.83}$$

Combining equation (8.82) with equation (8.83) results in

$$d\epsilon_{x'} - d\epsilon_{y'} = d\gamma_{xy} \tag{8.84}$$

Equation (8.84) describing in Lagrangian coordinates a single particle. Changing it to the Eulerian coordinates and location differential transform equation (8.84) into

$$\frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{D\gamma_{xy}}{Dt} \tag{8.85}$$

From (8.69) it can be observed that the right hand side can be replaced by $\tau_{xy}/\mu$.

$$\frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{\tau_{xy}}{\mu} \tag{8.86}$$

From equation (8.76) $\tau_{xy}$ be substituted and equation (8.86) can be continued and replaced as

$$\frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{1}{2\,\mu}\left(\tau_{x'x'} - \tau_{y'y'}\right) \tag{8.87}$$

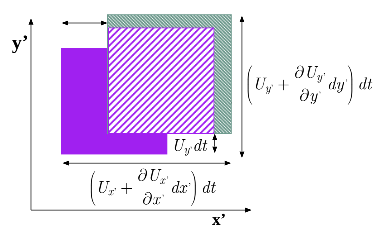



*Fig. -8.12. Linear strain of the element purple denotes $t$ and blue is for $t + dt$. Dashed squares denotes the movement without the linear change.*

Figure 8.12 depicts the approximate linear deformation of the element. The linear deformation is the difference between the two sides as

$$\frac{D\epsilon_{x'}}{Dt} = \frac{\partial U_{x'}}{\partial x'} \tag{8.88}$$

The same way it can written for the $y'$ coordinate.

$$\frac{D\epsilon_{y'}}{Dt} = \frac{\partial U_{y'}}{\partial y'} \tag{8.89}$$

Equation (8.88) can be written in the $y'$ and is similar by substituting the coordinates. The rate of strain relations can be substituted by the velocity and equations (8.88) and (8.89) changes into

$$\tau_{x'x'} - \tau_{y'y'} = 2\mu\left(\frac{\partial U_{x'}}{\partial x'} - \frac{\partial U_{y'}}{\partial y'}\right) \tag{8.90}$$

Similar two equations can be obtained in the other two plans. For example in $y'$–$z'$ plan one can obtained

$$\tau_{x'x'} - \tau_{z'z'} = 2\mu \left( \frac{\partial U_{x'}}{\partial x'} - \frac{\partial U_{z'}}{\partial z'} \right) \tag{8.91}$$

Adding equations (8.90) and (8.91) results in

$$\overbrace{(3-1)}^{2}\, \tau_{x'x'} - \tau_{y'y'} - \tau_{z'z'} = \overbrace{(6-2)}^{4}\, \mu \frac{\partial U_{x'}}{\partial x'} - 2\,\mu \left( \frac{\partial U_{y'}}{\partial y'} + \frac{\partial U_{z'}}{\partial z'} \right) \tag{8.92}$$

rearranging equation (8.92) transforms it into

$$3\,\tau_{x'x'} = \tau_{x'x'} + \tau_{y'y'} + \tau_{z'z'} + 6\,\mu \frac{\partial U_{x'}}{\partial x'} - 2\,\mu \left( \frac{\partial U_{x'}}{\partial x'} + \frac{\partial U_{y'}}{\partial y'} + \frac{\partial U_{z'}}{\partial z'} \right) \tag{8.93}$$

Dividing the restuls by 3 so that one can obtained the following

$$\tau_{x'x'} = \overbrace{\frac{\tau_{x'x'} + \tau_{y'y'} + \tau_{z'z'}}{3}}^{\text{“}mechanical''pressure} + 2\,\mu \frac{\partial U_{x'}}{\partial x'} - \frac{2}{3}\,\mu \left( \frac{\partial U_{x'}}{\partial x'} + \frac{\partial U_{y'}}{\partial y'} + \frac{\partial U_{z'}}{\partial z'} \right) \tag{8.94}$$

The “mechanical” pressure, $P_m$, is defined as the (negative) average value of pressure in directions of $x'$–$y'$–$z'$. This pressure is a true scalar value of the flow field since the propriety is averaged or almost[16] invariant to the coordinate transformation. In situations where the main diagonal terms of the stress tensor are not the same in all directions (in some viscous flows) this property can be served as a measure of the local normal stress. The mechanical pressure can be defined as averaging of the normal stress acting on a infinitesimal sphere. It can be shown that this two definitions are “identical” in the limits[17]. With this definition and noticing that the coordinate system $x'$– $y'$ has no special significance and hence equation (8.94) must be valid in any coordinate system thus equation (8.94) can be written as

$$\tau_{xx} = -P_m + 2\,\mu \frac{\partial U_x}{\partial x} + \frac{2}{3}\mu \nabla \cdot \boldsymbol{U} \tag{8.95}$$

Again where $P_m$ is the mechanical pressure and is defined as

Mechanical Pressure

$$P_m = -\frac{\tau_{xx} + \tau_{yy} + \tau_{zz}}{3} \tag{8.96}$$

It can be observed that the non main (diagonal) terms of the stress tensor are represented by an equation like (8.72). Commonality engineers like to combined the two difference

[16]It identical only in the limits to the mechanical measurements.

[17]G. K. Batchelor, An Introduction to Fluid Mechanics, Cambridge University Press, 1967, p.141.

expressions into one as

$$\tau_{xy} = -\left(P_m + \frac{2}{3}\mu\nabla\cdot\boldsymbol{U}\right)\overbrace{\delta_{xy}}^{=0} + \mu\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) \tag{8.97}$$

or

$$\tau_{xx} = -\left(P_m + \frac{2}{3}\mu\nabla\cdot\boldsymbol{U}\right)\overbrace{\delta_{xy}}^{=1} + \mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y}\right) \tag{8.98}$$

*Advance material can be skipped*

or index notation

$$\tau_{ij} = -\left(P_m + \frac{2}{3}\mu\nabla\cdot\boldsymbol{U}\right)\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{8.99}$$

*End Advance material*

where $\delta_{ij}$ is the Kronecker delta what is $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ otherwise. While this expression has the advantage of compact writing, it does not add any additional information. This expression suggests a new definition of the thermodynamical pressure is

Thermodynamic Pressure

$$P = P_m + \frac{2}{3}\mu\nabla\cdot\boldsymbol{U} \tag{8.100}$$

### Summary of The Stress Tensor

The above derivations were provided as a long mathematical explanation[18]. To reduced one unknown (the shear stress) equation (8.61) the relationship between the stress tensor and the velocity were to be established. First, connection between $\tau_{xy}$ and the deformation was built. Then the association between normal stress and perpendicular stress was constructed. Using the coordinates transformation, this association was established. The linkage between the stress in the rotated coordinates to the deformation was established.

### Second Viscosity Coefficient

The coefficient $2/3\mu$ is experimental and relates to viscosity. However, if the derivations before were to include additional terms, an additional correction will be needed. This correction results in

$$P = P_m + \lambda\nabla\cdot\boldsymbol{U} \tag{8.101}$$

[18]Since the publishing the version 0.2.9.0 several people ask this author to summarize conceptually the issues. With God help, it will be provide before version 0.3.1

The value of $\lambda$ is obtained experimentally. This coefficient is referred in the literature by several terms such as the "expansion viscosity" "second coefficient of viscosity" and "bulk viscosity." Here the term bulk viscosity will be adapted. The dimension of the bulk viscosity, $\lambda$, is similar to the viscosity $\mu$. $\lambda$ bulk viscosity

According to second law of thermodynamic derivations (not shown here and are under construction) demonstrate that $\lambda$ must be positive. The thermodynamic pressure always tends to follow the mechanical pressure during a change. The expansion rate of change and the fluid molecular structure through $\lambda$ control the difference. Equation (8.101) can be written in terms of the thermodynamic pressure $P$, as

$$\tau_{ij} = -\left[P + \left(\frac{2}{3}\mu - \lambda\right)\nabla\cdot\boldsymbol{U}\right]\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{8.102}$$

The significance of the difference between the thermodynamic pressure and the mechanical pressure associated with fluid dilation which connected by $\nabla\cdot\boldsymbol{U}$. The physical meaning of $\nabla\cdot\boldsymbol{U}$ represents the relative volume rate of change. For simple gas (dilute monatomic gases) it can be shown that $\lambda$ vanishes. In material such as water, $\lambda$ is large (3 times $\mu$) but the net effect is small because in that cases $\nabla\cdot\boldsymbol{U} \longrightarrow 0$. For complex liquids this coefficient, $\lambda$, can be over 100 times larger than $\mu$. Clearly for incompressible flow, this coefficient or the whole effect is vanished[19]. In most cases, the total effect of the dilation on the flow is very small. Only in micro fluids and small and molecular scale such as in shock waves this effect has some significance. In fact this effect is so insignificant that there is difficulty in to construct experiments so this effect can be measured. Thus, neglecting this effect results in

$$\tau_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{8.103}$$

To explain equation (8.103), it can be written for spesific coordinates. For example, for the $\tau_{xx}$ it can be written that

$$\tau_{xx} = -P + 2\frac{\partial U_x}{\partial x} \tag{8.104}$$

and the $y$ coordinate the equation is

$$\tau_{yy} = -P + 2\frac{\partial U_y}{\partial y} \tag{8.105}$$

however the mix stress, $\tau_{xy}$, is

$$\tau_{xy} = \tau_{yx} = \left(\frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y}\right) \tag{8.106}$$

[19]The reason that the effect vanish is because $\nabla\cdot\boldsymbol{U} = 0$.

For the total effect, substitute equation (8.102) into equation (8.61) which results in

$$\rho\left(\frac{DU_x}{Dt}\right) = -\frac{\partial\left(P + \left(\frac{2}{3}\mu - \lambda\right)\nabla\cdot\boldsymbol{U}\right)}{\partial x} + \mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right) \quad + \boldsymbol{f}_{Bx} \tag{8.107}$$

or in a vector form as

$$\rho\,\frac{D\boldsymbol{U}}{Dt} = -\nabla P + \left(\frac{1}{3}\mu + \lambda\right)\nabla\,(\nabla\cdot\boldsymbol{U}) + \mu\,\nabla^2\boldsymbol{U} + \boldsymbol{f}_B \tag{8.108}$$

Por in index form as

$$\rho\,\frac{D\,U_i}{Dt} = -\frac{\partial}{\partial x_i}\left(P + \left(\frac{2}{3}\mu - \lambda\right)\nabla\cdot\boldsymbol{U}\right) + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right) + \boldsymbol{f}_{Bi} \tag{8.109}$$

For incompressible flow the term $\nabla\cdot\boldsymbol{U}$ vanishes, thus equation (8.108) is reduced to

**Momentum for Incompressible Flow**

$$\rho\,\frac{D\boldsymbol{U}}{Dt} = -\nabla P + \mu\,\nabla^2\boldsymbol{U} + \boldsymbol{f}_B \tag{8.110}$$

or in the index notation it is written

$$\rho\,\frac{D\,U_i}{Dt} = -\frac{\partial P}{\partial x_i} + \mu\,\frac{\partial^2 \boldsymbol{U}}{\partial x_i \partial x_j} + \boldsymbol{f}_{Bi} \tag{8.111}$$

The momentum equation in Cartesian coordinate can be written explicitly for $x$ coordinate as

$$\begin{aligned}\rho\left(\frac{\partial U_x}{\partial t} + \right. & \left. U_x\frac{\partial U_x}{\partial x} + U_y\frac{\partial U_y}{\partial y} + U_z\frac{\partial U_z}{\partial z}\right) = \\ & -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right) + \rho g_x\end{aligned} \tag{8.112}$$

Where $g_x$ is the the body force in the $x$ direction $(\widehat{i}\cdot\boldsymbol{g})$. In the $y$ coordinate the momentum equation is

$$\begin{aligned}\rho\left(\frac{\partial U_y}{\partial t} + \right. & \left. U_x\frac{\partial U_y}{\partial x} + U_y\frac{\partial U_y}{\partial y} + U_z\frac{\partial U_z}{\partial z}\right) = \\ & -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y\end{aligned} \tag{8.113}$$

in $z$ coordinate the momentum equation is

$$\begin{aligned}\rho\left(\frac{\partial U_z}{\partial t} + \right. & \left. U_x\frac{\partial U_z}{\partial x} + U_y\frac{\partial U_z}{\partial y} + U_z\frac{\partial U_z}{\partial z}\right) = \\ & -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 U_z}{\partial x^2} + \frac{\partial^2 U_z}{\partial y^2} + \frac{\partial^2 U_z}{\partial z^2}\right) + \rho g_z\end{aligned} \tag{8.114}$$

# 8.6 Boundary Conditions and Driving Forces

## 8.6.1 Boundary Conditions Categories

The governing equations that were developed earlier requires some boundary conditions and initial conditions. These conditions described physical situations that are believed or should exist or approximated. These conditions can be categorized by the velocity, pressure, or in more general terms as the shear stress conditions (mostly at the interface). For this discussion, the shear tensor will be separated into two categories, pressure (at the interface direction) and shear stress (perpendicular to the area). A common velocity condition is that the liquid has the same value as the solid interface velocity. In the literature, this condition is referred as the "no slip" condition. The solid surface is rough thus the liquid participles (or molecules) are slowed to be at the solid surface velocity. This boundary condition was experimentally observed under many conditions yet it is not universal true. The slip condition (as oppose to "no slip" condition) exist in situations where the scale is very small and the velocity is relatively very small. The slip condition is dealing with a difference in the velocity between the solid (or other material) and the fluid media. The difference between the small scale and the large scale is that the slip can be neglected in the large scale while the slip cannot be neglected in the small scale. In another view, the difference in the velocities vanishes as the scale increases. Another condition which affects whether the slip condition exist is how rapidly of the velocity change. The slip condition cannot be ignored in some regions, when the flow is with a strong velocity fluctuations. Mathematically the "no slip" condition is written as

$$\widehat{\mathbf{t}} \cdot (\boldsymbol{U}_{fluid} - \boldsymbol{U}_{boundary}) = 0 \tag{8.115}$$

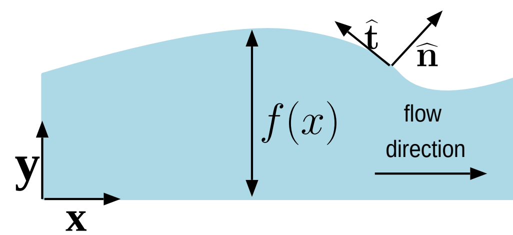



*Fig. -8.13. 1–Dimensional free surface describing $\widehat{\boldsymbol{n}}$ and $\widehat{\boldsymbol{t}}$.*

where $\widehat{\mathbf{n}}$ is referred to the area direction (perpendicular to the area see Figure 8.13). While this condition (8.115) is given in a vector form, it is more common to write this condition as a given velocity at a certain point such as

$$U(\ell) = U_\ell \tag{8.116}$$

Note, the "no slip" condition is applicable to the ideal fluid ("inviscid flows") because this kind of flow normally deals with large scales. The "slip" condition is written in similar fashion to equation (8.115) as

$$\widehat{\mathbf{t}} \cdot (\boldsymbol{U}_{fluid} - \boldsymbol{U}_{boundary}) = f(Q, scale, etc) \tag{8.117}$$

As oppose to a given velocity at particular point, a requirement on the acceleration (velocity) can be given in unknown position. The condition (8.115) can be mathematically represented in another way for free surface conditions. To make sure that all the material is accounted for in the control volume (does not cross the free surface), the relative perpendicular velocity at the interface must be zero. The location of the (free) moving boundary can be given as $f(\widehat{\boldsymbol{r}}, t) = 0$ as the equation which describes the bounding surface. The perpendicular relative velocity at the surface must be zero and therefore

$$\frac{Df}{Dt} = 0 \quad \text{on the surface } f(\widehat{\boldsymbol{r}}, t) = 0 \tag{8.118}$$

This condition is called the kinematic boundary condition. For example, the free surface in the two dimensional case is represented as $f(t, x, y)$. The condition becomes as

$$0 = \frac{\partial f}{\partial t} + U_x \frac{\partial f}{\partial x} + U_y \frac{\partial f}{\partial y} \tag{8.119}$$

The solution of this condition, sometime, is extremely hard to handle because the location isn't given but the derivative given on unknown location. In this book, this condition will not be discussed (at least not plane to be written).

The free surface is a special case of moving surfaces where the surface between two distinct fluids. In reality the interface between these two fluids is not a sharp transition but only approximation (see for the surface theory). There are situations where the transition should be analyzed as a continuous transition between two phases. In other cases, the transition is idealized an almost jump (a few molecules thickness). Furthermore, there are situations where the fluid (above one of the sides) should be considered as weightless material. In these cases the assumptions are that the transition occurs in a sharp line, and the density has a jump while the shear stress are continuous (in some cases continuously approach zero value). While a jump in density does not break any physical laws (at least those present in the solution), the jump in a shear stress (without a jump in density) does break a physical law. A jump in the shear stress creates infinite force on the adjoin thin layer. Off course, this condition cannot be tolerated since infinite velocity (acceleration) is impossible. The jump in shear stress can appear when the density has a jump in density. The jump in the density (between the two fluids) creates a surface tension which offset the jump in the shear stress. This condition is expressed mathematically by equating the shear stress difference to the forces results due to the surface tension. The shear stress difference is

$$\Delta\boldsymbol{\tau}^{(n)} = 0 = \Delta\boldsymbol{\tau}^{(n)}{}_{\substack{\text{upper}\\\text{surface}}} - \Delta\boldsymbol{\tau}^{(n)}{}_{\substack{\text{lower}\\\text{surface}}} \tag{8.120}$$

where the index $(n)$ indicate that shear stress are normal (in the surface area). If the surface is straight there is no jump in the shear stress. The condition with curved surface are out the scope of this book yet mathematically the condition is given as without explanation as

$$\widehat{\boldsymbol{n}} \cdot \boldsymbol{\tau}^{(n)} = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \tag{8.121}$$

$$\widehat{\boldsymbol{t}} \cdot \boldsymbol{\tau}^{(t)} = -\widehat{\boldsymbol{t}} \cdot \nabla \sigma \tag{8.122}$$

where $\widehat{\boldsymbol{n}}$ is the unit normal and $\widehat{\boldsymbol{t}}$ is a unit tangent to the surface (notice that direction pointed out of the "center" see Figure 8.13) and $R_1$ and $R_2$ are principal radii. One of results of the free surface condition (or in general, the moving surface condition) is that integration constant is unknown). In same instances, this constant is determined from the volume conservation. In index notation equation (8.121) is written[20] as

$$\tau_{ij}^{(1)}\, n_j + \sigma\, n_i \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = \tau_{ij}^{(2)}\, n_j \tag{8.123}$$

where $1$ is the upper surface and $2$ is the lower surface. For example in one dimensional[21]

$$\begin{aligned} \widehat{\boldsymbol{n}} &= \frac{(-f'(x), 1)}{\sqrt{1 + (f'(x))^2}} \\ \widehat{\boldsymbol{t}} &= \frac{(1, f'(x))}{\sqrt{1 + (f'(x))^2}} \end{aligned} \tag{8.124}$$

the unit vector is given as two vectors in $x$ and $y$ and the radius is given by equation (1.57). The equation is given by

$$\frac{\partial f}{\partial t} + U_x \frac{\partial f}{\partial x} = U_y \tag{8.125}$$

#### The Pressure Condition

The second condition that commonality prescribed at the interface is the static pressure at a specific location. The static pressure is measured perpendicular to the flow direction. The last condition is similar to the pressure condition of prescribed shear stress or a relationship to it. In this category include the boundary conditions with issues of surface tension which were discussed earlier. It can be noticed that the boundary conditions that involve the surface tension are of the kind where the condition is given on boundary but no at a specific location.

#### Gravity as Driving Force

[20]There is no additional benefit in this writing, it just for completeness and can be ignored for most purposes.

[21]A one example of a reference not in particularly important or significant just a random example. Jean, M. Free surface of the steady flow of a Newtonian fluid in a finite channel. Arch. Rational Mech. Anal. 74 (1980), no. 3, 197–217.

The body forces, in general and gravity in a particular, are the condition that given on the flow beside the velocity, shear stress (including the surface tension) and the pressure. The gravity is a common body force which is considered in many fluid mechanics problems. The gravity can be considered as a constant force in most cases (see for dimensional analysis for the reasons).

**Shear Stress and Surface Tension as Driving Force**

If the fluid was solid material, pulling the side will pull all the material. In fluid (mostly liquid) shear stress pulling side (surface) will have limited effect and yet sometime is significant and more rarely dominate. Consider, for example, the case shown in Figure 8.14. The shear stress carry the material as if part of the material was a solid material. For example, in the kerosene lamp the burning occurs at the surface of the lamp top and the liquid is at the bottom. The liquid does not move up due the gravity (actually it is against the gravity) but because the surface tension.

*Fig. -8.14. Kerosene lamp.*

The physical conditions in Figure 8.14 are used to idealize the flow around an inner rode to understand how to apply the surface tension to the boundary conditions. The fluid surrounds the rode and flows upwards. In that case, the velocity at the surface of the inner rode is zero. The velocity at the outer surface is unknown. The boundary condition at outer surface given by a jump of the shear stress. The outer diameter is depends on the surface tension (the larger surface tension the smaller the liquid diameter). The surface tension is a function of the temperature therefore the gradient in surface tension is result of temperature gradient. In this book, this effect is not discussed. However, somewhere downstream the temperature gradient is insignificant. Even in that case, the surface tension gradient remains. It can be noticed that, under the assumption presented here, there are two principal radii of the flow. One radius toward the center of the rode while the other radius is infinite (approximatly). In that case, the contribution due to the curvature is zero in the direction of the flow (see Figure 8.15). The only (almost) propelling source of the flow is the surface gradient $(\frac{\partial \sigma}{\partial n})$.

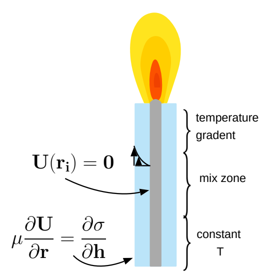



*Fig. -8.15. Schematic of kerosene lamp.*

## 8.7 Examples for Differential Equation (Navier-Stokes)

Examples of an one-dimensional flow driven by the shear stress and pressure are presented. For further enhance the understanding some of the derivations are repeated.

First, example dealing with one phase are present. Later, examples with two phase are presented.

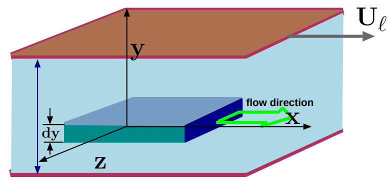



*Fig. -8.16. Flow between two plates, top plate is moving at speed of $U_\ell$ to the right (as positive). The control volume shown in darker colors.*

### Example 8.6:

*Incompressible liquid flows between two infinite plates from the left to the right (as shown in Figure 8.16). The distance between the plates is $\ell$. The static pressure per length is given as $\Delta P$[22]. The upper surface is moving in velocity, $U_\ell$ (The rightside is defined as positive).*

<u>Solution</u>

In this example, the mass conservation yields

$$\overbrace{\frac{d}{dt}\int_{cv}\rho dV}^{=0} = -\int_{cv}\rho\, U_{rn} dA = 0 \tag{8.126}$$

The momentum is not accumulated (steady state and constant density). Further because no change of the momentum thus

$$\int_{A}\rho\, U_x\, U_{rn} dA = 0 \tag{8.127}$$

Thus, the flow in and the flow out are equal. It can concluded that the velocity in and out are the same (for constant density). The momentum conservation leads

$$-\int_{cv}\boldsymbol{P} dA + \int_{cv}\boldsymbol{\tau}_{xy} dA = 0 \tag{8.128}$$

The reaction of the shear stress on the lower surface of control volume based on Newtonian fluid is

$$\boldsymbol{\tau}_{xy} = -\mu\frac{dU}{dy} \tag{8.129}$$

[22]The difference is measured at the bottom point of the plate.

On the upper surface is different by Taylor explanation as

$$\boldsymbol{\tau}_{xy} = \mu \left( \frac{dU}{dy} + \frac{d^2U}{dy^2}\,dy + \overbrace{\frac{d^3U}{dy^3}\,dy^2 + \cdots}^{\cong 0} \right) \tag{8.130}$$

The net effect of these two will be difference between them

$$\mu \left( \frac{dU}{dy} + \frac{d^2U}{dy^2}\,dy \right) - \mu \frac{dU}{dy} \cong \mu \frac{d^2U}{dy^2}\,dy \tag{8.131}$$

The assumptions is that there is no pressure difference in the $z$ direction. The only difference in the pressure is in the $x$ direction and thus

$$P - \left( P + \frac{dP}{dx}\,dx \right) = -\frac{dP}{dx}\,dx \tag{8.132}$$

A discussion why $\frac{\partial P}{\partial y} \sim 0$ will be presented later. The momentum equation in the $x$ direction (or from equation (8.112)) results (without gravity effects) in

$$-\frac{dP}{dx} = \mu \frac{d^2U}{dy^2} \tag{8.133}$$

Equation (8.133) was constructed under several assumptions which include the direction of the flow, Newtonian fluid. No assumption was imposed on the pressure distribution. Equation (8.133) is a partial differential equation but can be treated as ordinary differential equation in the $z$ direction of the pressure difference is uniform. In that case, the left hand side is equal to constant. The "standard" boundary conditions is non–vanishing pressure gradient (that is the pressure exist) and velocity of the upper or lower surface or both. It is common to assume that the "no slip" condition on the boundaries condition[23]. The boundaries conditions are

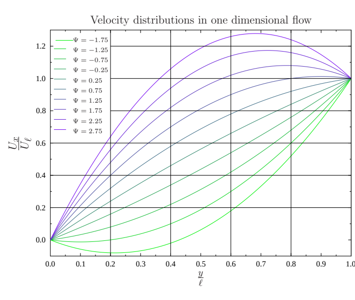



*Fig. -8.17. One dimensional flow with a shear between two plates when $\Psi$ change value between -1.75 green line to 3 the blue line.*

$$\begin{aligned} U_x(y = 0) &= 0 \\ U_x(y = \ell) &= U_\ell \end{aligned} \tag{8.134}$$

[23]A discussion about the boundary will be presented later.

The solution of the "ordinary" differential equation (8.133) after the integration becomes

$$U_x = -\frac{1}{2}\frac{dP}{dx}\,y^2 + c_2\,y + c_3 \tag{8.135}$$

Applying the boundary conditions, equation (8.134)/ results in

$$U_x(y) = \frac{y}{\ell}\left(\overbrace{\frac{\ell^2}{U_0\,2\mu}\frac{dP}{dx}}^{=\Psi}\left(1-\frac{y}{\ell}\right)\right)+\frac{y}{\ell} \tag{8.136}$$

For the case where the pressure gradient is zero the velocity is linear as was discussed earlier in Chapter 1 (see Figure 8.17). However, if the plates or the boundary conditions do not move the solution is

$$U_x(y) = \left(\frac{\ell^2}{U_0\,2\mu}\frac{dP}{dx}\left(1-\frac{y}{\ell}\right)\right)+\frac{y}{\ell} \tag{8.137}$$

What happen when $\frac{\partial P}{\partial y} \sim 0$?

End Solution

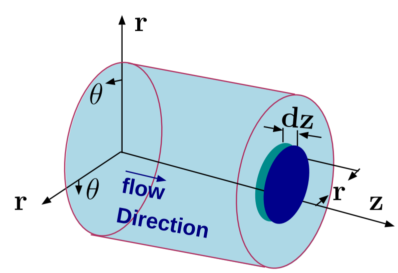



*Fig. -8.18. The control volume of liquid element in cylindrical coordinates.*

### Cylindrical Coordinates

Similarly the problem of one dimensional flow can be constructed for cylindrical coordinates. The problem is still one dimensional because the flow velocity is a function of (only) radius. This flow referred as Poiseuille flow after Jean Louis Poiseuille a French Physician who investigated blood flow in veins. Thus, Poiseuille studied the flow in a small diameters (he was not familiar with the concept of Reynolds numbers). Rederivation are carried out for a short cut.

The momentum equation for the control volume depicted in the Figure 8.18a is

$$-\int \boldsymbol{P}\,dA + \int \boldsymbol{\tau} dA = \int \rho\,U_z\,U_{rn}\,dA \tag{8.138}$$

The shear stress in the front and back surfaces do no act in the $z$ direction. The shear stress on the circumferential part small dark blue shown in Figure 8.18a is

$$\int \boldsymbol{\tau}\, dA = \mu \frac{dU_z}{dr} \overbrace{2\,\pi\,r\,dz}^{dA} \tag{8.139}$$

The pressure integral is

$$\int \boldsymbol{P}\, dA = \left(P_{z_dz} - P_z\right)\,\pi\,r^2 = \left(P_z + \frac{\partial P}{\partial z} dz - P_z\right)\,\pi\,r^2 = \frac{\partial P}{\partial z} dz\,\pi\,r^2 \tag{8.140}$$

The last term is

$$\begin{aligned} &\int \rho\, U_z\, U_{rn}\, dA = \rho \int \quad U_z\, U_{rn}\, dA = \\ &\rho\left(\int_{z+dz} {U_{z+dz}}^2 dA \quad - \int_z {U_z}^2 dA\right) = \rho \int_z \left({U_{z+dz}}^2 - {U_z}^2\right) dA \end{aligned} \tag{8.141}$$

The term ${U_{z+dz}}^2 - {U_z}^2$ is zero because $U_{z+dz} = U_z$ because mass conservation conservation for any element. Hence, the last term is

$$\int \rho\, U_z\, U_{rn}\, dA = 0 \tag{8.142}$$

Substituting equation (8.139) and (8.140) into equation (8.138) results in

$$\mu \frac{dU_z}{dr} 2 \cancel{\pi} \cancel{r} \cancel{dz} = -\frac{\partial P}{\partial z} \cancel{dz} \cancel{\pi} r^{\cancel{2}} \tag{8.143}$$

Which shrinks to

$$\frac{2\,\mu}{r}\, \frac{dU_z}{dr} = -\frac{\partial P}{\partial z} \tag{8.144}$$

Equation (8.144) is a first order differential equation for which only one boundary condition is needed. The "no slip" condition is assumed

$$U_z(r = R) = 0 \tag{8.145}$$

Where $R$ is the outer radius of pipe or cylinder. Integrating equation (8.144) results in

$$U_z = -\frac{1}{\mu} \frac{\partial P}{\partial z}\, r^2 + c_1 \tag{8.146}$$

It can be noticed that asymmetrical element[24] was eliminated due to the smart short cut. The integration constant obtained via the application of the boundary condition which is

$$c_1 = -\frac{1}{\mu} \frac{\partial P}{\partial z}\, R^2 \tag{8.147}$$

[24] Asymmetrical element or function is $-f(x) = f(-x)$

The solution is

$$U_z = \frac{1}{\mu}\frac{\partial P}{\partial z} R^2 \left(1 - \left(\frac{r}{R}\right)^2\right) \tag{8.148}$$

While the above analysis provides a solution, it has several deficiencies which include the ability to incorporate different boundary conditions such as flow between concentering cyliders.

### Example 8.7:

*A liquid with a constant density is flowing between concentering cylinders as shown in Figure 8.19. Assume that the velocity at the surface of the cylinders is zero calculate the velocity profile. Build the velocity profile when the flow is one directional and viscosity is Newtonian. Calculate the flow rate for a given pressure gradient.*

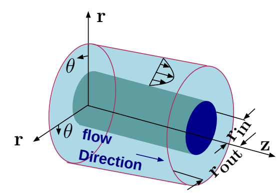



*Fig. -8.19. Liquid flow between concentric cylinders for example 8.7.*

SOLUTION

After the previous example, the appropriate version of the Navier–Stokes equation will be used. The situation is best suitable to solved in cylindrical coordinates. One of the solution of this problems is one dimensional solution. In fact there is no physical reason why the flow should be only one dimensional. However, it is possible to satisfy the boundary conditions. It turn out that the “simple” solution is the first mode that appear in reality. In this solution will be discussing the flow first mode. For this mode, the flow is assumed to be one dimensional. That is, the velocity isn't a function of the angle, or $z$ coordinate. Thus only equation in $z$ coordinate is needed. It can be noticed that this case is steady state and also the acceleration (convective acceleration) is zero

$$\rho\left(\overbrace{\frac{\partial U_z}{\partial t}}^{\neq f(t)} + \overbrace{U_r}^{=0}\frac{\partial U_z}{\partial r} + \overbrace{\frac{U_\phi}{r}}^{=0}\overbrace{\frac{\partial U_z}{\partial \phi}}^{U_z \neq f(\phi)} + U_z\overbrace{\frac{\partial U_z}{\partial z}}^{=0}\right) = 0 \tag{8.149}$$

The steady state governing equation then becames

$$\rho\left(\emptyset\right) = 0 = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_z}{\partial r}\right) + \overbrace{\cdots}^{=0}\right) + \cancel{\rho g_z} \tag{8.VII.a}$$

The PDE above (8.VII.a) required boundary conditions which are

$$\begin{aligned} U_z\left(r = r_i\right) =& \quad 0 \\ U_z\left(r = r_o\right) =& \quad 0 \end{aligned} \tag{8.VII.b}$$

Integrating equation (8.VII.a) once results in

$$r\frac{\partial U_z}{\partial r} = \frac{1}{2\,\mu}\frac{\partial P}{\partial z}r^2 + c_1 \tag{8.VII.c}$$

Dividing equation (8.VII.c) and integrating results for the second times results

$$\frac{\partial U_z}{\partial r} = \frac{1}{2\,\mu}\frac{\partial P}{\partial z}r + \frac{c_1}{r} \tag{8.VII.d}$$

Integration of equation (8.VII.d) results in

$$U_z = \frac{1}{4\,\mu}\frac{\partial P}{\partial z}r^2 + c_1\,\ln r + c_2 \tag{8.VII.e}$$

Applying the first boundary condition results in

$$0 = \frac{1}{4\,\mu}\frac{\partial P}{\partial z}{r_i}^2 + c_1\,\ln r_i + c_2 \tag{8.VII.f}$$

applying the second boundary condition yields

$$0 = \frac{1}{4\,\mu}\frac{\partial P}{\partial z}{r_o}^2 + c_1\,\ln r_o + c_2 \tag{8.VII.g}$$

The solution is

$$\begin{aligned} c_1 &= \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left({r_o}^2 - {r_i}^2\right) \\ c_2 &= \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left(\ln(r_i)\,{r_o}^2 - \ln(r_o)\,{r_i}^2\right) \end{aligned} \tag{8.VII.h}$$

The solution is when substituting the constats into equation (8.VII.e) results in

$$\begin{aligned} U_z(r) = \frac{1}{4\,\mu}\frac{\partial P}{\partial z}r^2 + \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left({r_o}^2 - {r_i}^2\right)\ln r \\ + \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left(\ln(r_i)\,{r_o}^2 - \ln(r_o)\,{r_i}^2\right) \end{aligned} \tag{8.VII.i}$$

The flow rate is then

$$Q = \int_{r_i}^{r_o} U_z(r)dA \tag{8.VII.j}$$

Or substituting equation (8.VII.i) into equation (8.VII.j) transfomed into

$$\begin{aligned} Q = \int_A \left[\frac{1}{4\,\mu}\frac{\partial P}{\partial z}r^2 + \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left({r_o}^2 - {r_i}^2\right)\ln r \right. \\ \left. + \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left(\ln(r_i)\,{r_o}^2 - \ln(r_o)\,{r_i}^2\right)\right] dA \end{aligned} \tag{8.VII.k}$$

A finite intergation of the last term in the integrand results in zero because it is constant. The integraion of the rest is

$$Q = \left[\frac{1}{4\,\mu}\frac{\partial P}{\partial z}\right]\int_{r_i}^{r_o}\left[r^2 + \ln\left(\frac{r_o}{r_i}\right)\left({r_o}^2 - {r_i}^2\right)\ln\,r\right]2\,\pi\,r\,dr \tag{8.VII.l}$$

The first integration of the first part of the second squere bracket, $(r^3)$, is $1/4\left({r_o}^4 - {r_i}^4\right)$. The second part, of the second squere bracket, $(-a \times r \ln\, r)$ can be done by parts to be as

$$a\left(\frac{r^2}{4} - \frac{r^2\log(r)}{2}\right)$$

Applying all these "techniques" to equation (8.VII.l) results in

$$\begin{aligned} Q = &\left[\frac{\pi}{2\,\mu}\frac{\partial P}{\partial z}\right]\left[\left(\frac{{r_o}^4}{4} - \frac{{r_i}^4}{4}\right) + \right.\\ \ln\left(\frac{r_o}{r_i}\right) &\left.\left({r_o}^2 - {r_i}^2\right)\left(\frac{{r_o}^2\ln(r_o)}{2} - \frac{{r_o}^2}{4} - \frac{{r_i}^2\ln(r_i)}{2} + \frac{{r_i}^2}{4}\right)\right] \end{aligned} \tag{8.VII.m}$$

The averaged velocity is obtained by dividing flow rate by the area $Q/A$.

$$U_{ave} = \frac{Q}{\pi\left({r_o}^2 - {r_i}^2\right)} \tag{8.150}$$

in which the identy of $(a^4 - b^4)/(a^2 - b^2)$ is $b^2 + a^2$ and hence

$$\begin{aligned} U_{ave} = &\left[\frac{1}{2\,\mu}\frac{\partial P}{\partial z}\right]\left[\left(\frac{{r_o}^2}{4} + \frac{{r_i}^2}{4}\right) + \right.\\ \ln\left(\frac{r_o}{r_i}\right) &\left.\left(\frac{{r_o}^2\ln(r_o)}{2} - \frac{{r_o}^2}{4} - \frac{{r_i}^2\ln(r_i)}{2} + \frac{{r_i}^2}{4}\right)\right] \end{aligned} \tag{8.VII.n}$$

End Solution

The next example deals with the gravity as body force in two dimensional flow. This problem study by Nusselt[25] which developed the basics equations. This problem is related to many industrial process and is fundamental in understanding many industrial processes. Furthermore, this analysis is a building bloc for heat and mass transfer understanding[26].

Example 8.8:

*In many situations in nature and many industrial processes liquid flows downstream*

---

[25] German mechanical engineer, Ernst Kraft Wilhelm Nusselt born November 25, 1882 September 1, 1957 in Munchen

[26] Extensive discussion can be found in this author master thesis. Comprehensive discussion about this problem can be found this author Master thesis.

*on inclined plate at $\theta$ as shown in Figure 8.20. For this example, assume that the gas density is zero (located outside the liquid domain). Assume that "scale" is large enough so that the "no slip" condition prevail at the plate (bottom). For simplicity, assume that the flow is two dimensional. Assume that the flow obtains a steady state after some length (and the acceleration vanished). The dominate force is the gravity. Write the governing equations for this situation. Calculate the velocity profile. Assume that the flow is one dimensional in the $x$ direction.*

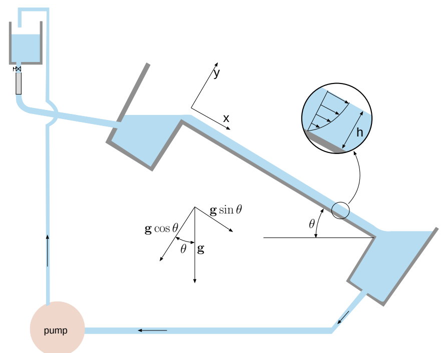



*Fig. -8.20. Mass flow due to temperature difference for example 8.1*

Solution

This problem is satiable to Cartesian coordinates in which $x$ coordinate is pointed in the flow direction and $y$ perpendicular to flow direction (depicted in Figure 8.20). For this system, the gravity in the $x$ direction is $g\sin\theta$ while the direction of $y$ the gravity is $g\cos\theta$. The governing in the $x$ direction is

$$\rho\left(\overbrace{\frac{\partial U_x}{\partial t}}^{\neq f(t)} + U_x\overbrace{\frac{\partial U_x}{\partial x}}^{=0} + \overbrace{U_y}^{=0}\frac{\partial U_y}{\partial y} + \overbrace{U_z}^{-0}\frac{\partial U_z}{\partial z}\right) = \\ -\overbrace{\frac{\partial P}{\partial x}}^{\sim 0} + \mu\left(\overbrace{\frac{\partial^2 U_x}{\partial x^2}}^{=0} + \frac{\partial^2 U_x}{\partial y^2} + \overbrace{\frac{\partial^2 U_x}{\partial z^2}}^{=0}\right) + \rho\overbrace{g_x}^{g\sin\theta} \tag{8.VIII.a}$$

The first term of the acceleration is zero because the flow is in a steady state. The first term of the convective acceleration is zero under the assumption of this example flow is fully developed and hence not a function of $x$ (nothing to be "improved"). The second and the third terms in the convective acceleration are zero because the velocity at that direction is zero ($U_y = U_z = 0$). The pressure is almost constant along the $x$ coordinate. As it will be shown later, the pressure loss in the gas phase (mostly air) is negligible. Hence the pressure at the gas phase is almost constant hence the pressure at the interface in the liquid is constant. The surface has no curvature and hence the pressure at liquid side similar to the gas phase and the only change in liquid is in the $y$ direction. Fully developed flow means that the first term of the velocity Laplacian is zero ($\frac{\partial U_x}{\partial x} \equiv 0$). The last term of the velocity Laplacian is zero because no velocity in the $z$ direction.

Thus, equation (8.VIII.a) is reduced to

$$0 = \mu \frac{\partial^2 U_x}{\partial y^2} + \rho\, g \sin\theta \tag{8.VIII.b}$$

With boundary condition of "no slip" at the bottom because the large scale and steady state

$$U_x(y=0) = 0 \tag{8.VIII.c}$$

The boundary at the interface is simplified to be

$$\left.\frac{\partial U_x}{\partial y}\right|_{y=0} = \tau_{air}\,(\sim 0) \tag{8.VIII.d}$$

If there is additional requirement, such a specific velocity at the surface, the governing equation can not be sufficient from the mathematical point of view. Integration of equation (8.VIII.b) yields

$$\frac{\partial U_x}{\partial y} = \frac{\rho}{\mu}\, g \sin\theta\, y + c_1 \tag{8.VIII.e}$$

The integration constant can be obtain by applying the condition (8.VIII.d) as

$$\tau_{air} = \mu \left.\frac{\partial U_x}{\partial y}\right|_{h} = -\rho\, g \sin\theta \overbrace{h}^{y} + c_1\, \mu \tag{8.VIII.f}$$

Solving for $c_1$ results in

$$c_1 = \frac{\tau_{air}}{\mu} + \underbrace{\frac{1}{\nu}}_{\frac{\mu}{\rho}} g\, \sin\theta\, h \tag{8.VIII.g}$$

The second integration applying the second boundary condition yields $c_2 = 0$ results in

$$U_x = \frac{g\, \sin\theta}{\nu}\left(2\, y\, h - y^2\right) - \frac{\tau_{air}}{\mu} \tag{8.VIII.h}$$

When the shear stress caused by the air is neglected, the velocity profile is

$$U_x = \frac{g\, \sin\theta}{\nu}\left(2\, h\, y - y^2\right) \tag{8.VIII.i}$$

The flow rate per unit width is

$$\frac{Q}{W} = \int_A U_x dA = \int_0^h \left(\frac{g\, \sin\theta}{\nu}\left(2\, h\, y - y^2\right) - \frac{\tau_{air}}{\mu}\right) dy \tag{8.VIII.j}$$

Where $W$ here is the width into the page of the flow. Which results in

$$\frac{Q}{W} = \frac{g\, \sin\theta}{\nu}\frac{2\, h^3}{3} - \frac{\tau_{air}\, h}{\mu} \tag{8.VIII.k}$$

The average velocity is then

$$\overline{U_x} = \frac{\dfrac{Q}{W}}{h} = \frac{g\,\sin\theta}{\nu}\frac{2\,h^2}{3} - \frac{\tau_{air}}{\mu} \tag{8.VIII.l}$$

Note the shear stress at the interface can be positive or negative and hence can increase or decrease the flow rate and the averaged velocity.

End Solution

In the following following example the issue of driving force of the flow through curved interface is examined. The flow in the kerosene lamp is depends on the surface tension. The flow surface is curved and thus pressure is not equal on both sides of the interface.

Example 8.9:

*A simplified flow version the kerosene lump is of liquid moving up on a solid core. Assume that radios of the liquid and solid core are given and the flow is at steady state. Calculate the minimum shear stress that required to operate the lump (alternatively, the maximum height).*

### 8.7.1 Interfacial Instability

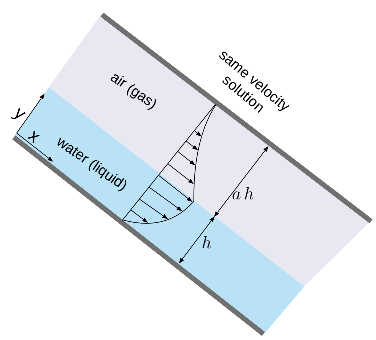



*Fig. -8.21. Flow of liquid in partially filled duct.*

In Example 8.8 no requirement was made as for the velocity at the interface (the upper boundary). The vanishing shear stress at the interface was the only requirement was applied. If the air is considered two governing equations must be solved one for the air (gas) phase and one for water (liquid) phase. Two boundary conditions must be satisfied at the interface. For the liquid, the boundary condition of "no slip" at the bottom surface of liquid must be satisfied. Thus, there is total of three boundary conditions[27] to be satisfied. The solution to the differential governing equations provides only two constants. The second domain (the gas phase) provides another equation with two constants but again three boundary conditions need to satisfied. However, two of the boundary conditions for these equations are the identical and thus the six boundary conditions are really only 4 boundary conditions.

[27] The author was hired to do experiments on thin film (gravity flow). These experiments were to study the formation of small and big waves at the interface. The phenomenon is explained by the fact that there is somewhere instability which is transferred into the flow. The experiments were conducted on a solid concrete laboratory and the flow was in a very stable system. No matter how low flow rate was small and big occurred. This explanation bothered this author, thus current explanation was developed to explain the wavy phenomenon occurs.

The governing equation solution[28] for the gas phase ($h \geq y \geq a\,h$) is

$$U_{x\,g} = \frac{g \sin\theta}{2\,\nu_g}\, y^2 + c_1\, y + c_2 \tag{8.151}$$

Note, the constants $c_1$ and $c_2$ are dimensional which mean that they have physical units ($c_1 \longrightarrow [1/sec]$ The governing equation in the liquid phase ($0 \geq y \geq h$) is

$$U_{x\ell} = \frac{g \sin\theta}{2\,\nu_\ell}\, y^2 + c_3\, y + c_4 \tag{8.152}$$

The gas velocity at the upper interface is vanished thus

$$U_{x\,g}\left[(1+a)\,h\right] = 0 \tag{8.153}$$

At the interface the "no slip" condition is regularly applied and thus

$$U_{x\,g}(h) = U_{x\ell}(h) \tag{8.154}$$

Also at the interface (a straight surface), the shear stress must be continuous

$$\mu_g \frac{\partial U_{x\,g}}{\partial y} = \mu_\ell \frac{\partial U_{x\ell}}{\partial y} \tag{8.155}$$

Assuming "no slip" for the liquid at the bottom boundary as

$$U_{x\ell}(0) = 0 \tag{8.156}$$

The boundary condition (8.153) results in

$$0 = \frac{g\,\sin\theta}{2\,\nu_g}\, h^2\,(1+a)^2 + c_1\, h\,(1+a) + c_2 \tag{8.157}$$

The same can be said for boundary condition (8.156) which leads

$$c_4 = 0 \tag{8.158}$$

Applying equation (8.155) yields

$$\overbrace{\frac{\mu_g}{\nu_g}}^{\rho_g}\; g\,\sin\theta\, h + c_1\,\mu_g = \overbrace{\frac{\mu_\ell}{\nu_\ell}}^{\rho_\ell}\; g\,\sin\theta\, h + c_3\,\mu_\ell \tag{8.159}$$

Combining boundary conditions equation(8.154) with (8.157) results in

$$\frac{g \sin\theta}{2\,\nu_g}\, h^2 + c_1\, h + c_2 = \frac{g \sin\theta}{2\,\nu_\ell}\, h^2 + c_3\, h \tag{8.160}$$

[28]This equation results from double integrating of equation (8.VIII.b) and subtitling $\nu = \mu/\rho$.

*Advance material can be skipped*

The solution of equation (8.157), (8.159) and (8.160) is obtained by computer algebra (see in the code) to be

$$
\begin{aligned}
c_1 &= -\frac{\sin\theta\,\left(g\,h\,\rho_g\,\left(2\,\rho_g\,\nu_\ell\,\rho_\ell+1\right)+a\,g\,h\,\nu_\ell\right)}{\rho_g\,\left(2\,a\,\nu_\ell+2\,\nu_\ell\right)} \\
c_2 &= \frac{\sin\theta\,\left(g\,h^2\,\rho_g\,\left(2\,\rho_g\,\nu_\ell\,\rho_\ell+1\right)-g\,h^2\,\nu_\ell\right)}{2\,\rho_g\,\nu_\ell} \\
c_3 &= \frac{\sin\theta\,\left(g\,h\,\rho_g\,\left(2\,a\,\rho_g\,\nu_\ell\,\rho_\ell-1\right)-a\,g\,h\,\nu_\ell\right)}{\rho_g\,\left(2\,a\,\nu_\ell+2\,\nu_\ell\right)}
\end{aligned}
\tag{8.161}
$$

*End Advance material*

When solving this kinds of mathematical problem the engineers reduce it to minimum amount of parameters to reduce the labor involve. So equation (8.157) transformed by some simple rearrangement to be

$$
\left(1+a\right)^2 = \overbrace{\frac{2\,\nu_g\,c_1}{g\,h\,\sin\theta}}^{C_1} + \overbrace{\frac{2\,c_2\,\nu_g}{g\,h^2\,\sin\theta}}^{C_2}
\tag{8.162}
$$

And equation (8.159)

$$
1+\overbrace{\frac{\nu_g\,c_1}{g\,h\,\sin\theta}}^{\frac{1}{2}\,C_1} = \frac{\rho_\ell}{\rho_g} + \overbrace{\frac{\mu_\ell\,\nu_g\,c_3}{\mu_g\,g\,h\,\sin\theta}}^{\frac{1}{2}\,\frac{\mu_\ell}{\mu_g}\,C_3}
\tag{8.163}
$$

and equation (8.160)

$$
1+\frac{2\,\nu_g\,\cancel{h}\,c_1}{h^{\cancel{2}}\,g\,\sin\theta} + \frac{2\,\nu_g\,c_2}{h^2\,g\,\sin\theta} = \frac{\nu_g}{\nu_\ell} + \frac{2\,\nu_g\,\cancel{h}\,c_3}{g\,h^{\cancel{2}}\,\sin\theta}
\tag{8.164}
$$

Or rearranging equation (8.164)

$$
\frac{\nu_g}{\nu_\ell}-1 = \overbrace{\frac{2\,\nu_g\,c_1}{h\,g\,\sin\theta}}^{C_1} + \overbrace{\frac{2\,\nu_g\,c_2}{h^2\,g\,\sin\theta}}^{C_2} - \overbrace{\frac{2\,\nu_g\,c_3}{g\,h\,\sin\theta}}^{C_3}
\tag{8.165}
$$

This presentation provide similarity and it will be shown in the Dimensional analysis chapter better physical understanding of the situation. Equation (8.162) can be written as

$$
\left(1+a\right)^2 = C_1 + C_2
\tag{8.166}
$$

Further rearranging equation (8.163)

$$\frac{\rho_\ell}{\rho_g} - 1 = \frac{C_1}{2} - \frac{\mu_\ell}{\mu_g}\frac{C_3}{2} \tag{8.167}$$

and equation (8.165)

$$\frac{\nu_g}{\nu_\ell} - 1 = C_1 + C_2 - C_3 \tag{8.168}$$

This process that was shown here is referred as non–dimensionalization[29]. The ratio of the dynamics viscosity can be eliminated from equation (8.168) to be

$$\frac{\mu_g}{\mu_\ell}\frac{\rho_\ell}{\rho_g} - 1 = C_1 + C_2 - C_3 \tag{8.169}$$

The set of equation can be solved for the any ratio of the density and dynamic viscosity. The solution for the constant is

$$C_1 = \frac{\rho_g}{\rho_\ell} - 2 + a^2 + 2\,a\frac{\mu_g}{\mu_\ell} + 2\frac{\mu_g}{\mu_\ell} \tag{8.170}$$

$$C_2 = \frac{-\dfrac{\mu_g}{\mu_\ell}\dfrac{\rho_\ell}{\rho_g} + a\left(2\dfrac{\mu_g}{\mu_\ell} - 2\right) + 3\dfrac{\mu_g}{\mu_\ell} + a^2\left(\dfrac{\mu_g}{\mu_\ell} - 1\right) - 2}{\dfrac{\mu_g}{\mu_\ell}} \tag{8.171}$$

$$C_3 = -\frac{\mu_g}{\mu_\ell}\frac{\rho_\ell}{\rho_g} + a^2 + 2\,a + 2 \tag{8.172}$$

The two different fluids[30] have flow have a solution as long as the distance is finite reasonable similar. What happen when the lighter fluid, mostly the gas, is infinite long. This is one of the source of the instability at the interface. The boundary conditions of flow with infinite depth is that flow at the interface is zero, flow at infinite is zero. The requirement of the shear stress in the infinite is zero as well. There is no way obtain one dimensional solution for such case and there is a component in the $y$ direction. Combining infinite size domain of one fluid with finite size on the other one side results in unstable interface.

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[29]Later it will be move to the Dimensional Chapter

[30]This topic will be covered in dimensional analysis in more extensively. The point here the understanding issue related to boundary condition not per se solution of the problem.

# CHAPTER 9

# Multi–Phase Flow

## 9.1 Introduction

Traditionally, the topic of multi–phase flow is ignored in an introductory class on fluid mechanics. For many engineers, this class will be the only opportunity to be exposed to this topic. The knowledge in this topic without any doubts, is required for many engineering problems. Calculations of many kinds of flow deals with more than one phase or material flow[1]. The author believes that the trends and effects of multiphase flow could and should be introduced and considered by engineers. In the past, books on multiphase flow were written more as a literature review or heavy on the mathematics. It is recognized that multiphase flow is still evolving. In fact, there is not a consensus to the exact map of many flow regimes. This book attempts to describe these issues as a fundamentals of physical aspects and less as a literature review. This chapter provides information that is more or less in consensus[2]. Additionally, the nature of multiphase flow requires solving many equations. Thus, in many books the representations is by writing the whole set governing equations. Here, it is believed that the interactions/calculations requires a full year class and hence, only the trends and simple calculations are described.

## 9.2 History

The study of multi–phase flow started for practical purposes after World War II. Initially the models were using simple assumptions. For simple models,there are two possibilities (1) the fluids/materials are flowing in well homogeneous mixed (where the main problem

[1]An example, there was a Ph.D. working for the government who analyzed filing cavity with liquid metal (aluminum), who did not consider the flow as two–phase flow and ignoring the air. As result, his analysis is in the twilight zone not in the real world.

[2]Or when the scientific principles simply dictate.

to find the viscosity), (2) the fluids/materials are flowing separately where the actual total loss pressure can be correlated based on the separate pressure loss of each of the material. If the pressure loss was linear then the total loss will be the summation of the two pressure losses (of the lighter liquid (gas) and the heavy liquid). Under this assumption the total is not linear and experimental correlation was made. The flow patterns or regimes were not considered. This was suggested by Lockhart and Martinelli who use a model where the flow of the two fluids are independent of each other. They postulate that there is a relationship between the pressure loss of a single phase and combine phases pressure loss as a function of the pressure loss of the other phase. It turned out this idea provides a good crude results in some cases.

Researchers that followed Lockhart and Martinelli looked for a different map for different combination of phases. When it became apparent that specific models were needed for different situations, researchers started to look for different flow regimes and provided different models. Also the researchers looked at the situation when the different regimes are applicable. Which leads to the concept of flow regime maps. Taitle and Duckler suggested a map based on five non-dimensional groups which are considered as the most useful today. However, Taitle and Duckler's map is not universal and it is only applied to certain liquid–gas conditions. For example, Taitle–Duckler's map is not applicable for microgravity.

## 9.3 What to Expect From This Chapter

As oppose to the tradition of the other chapters in this book and all other Potto project books, a description of what to expect in this chapter is provided. It is an attempt to explain and convince all the readers that the multi–phase flow must be included in introductory class on fluid mechanics[3]. Hence, this chapter will explain the core concepts of the multiphase flow and their relationship, and importance to real world.

This chapter will provide: a category of combination of phases, the concept of flow regimes, multi–phase flow parameters definitions, flow parameters effects on the flow regimes, partial discussion on speed of sound of different regimes, double choking phenomenon (hopefully), and calculation of pressure drop of simple homogeneous model. This chapter will introduce these concepts so that the engineer not only be able to understand a conversation on multi-phase but also, and more importantly, will know and understand the trends. However, this chapter will not provide a discussion of transient problems, phase change or transfer processes during flow, and actual calculation of pressure of the different regimes.

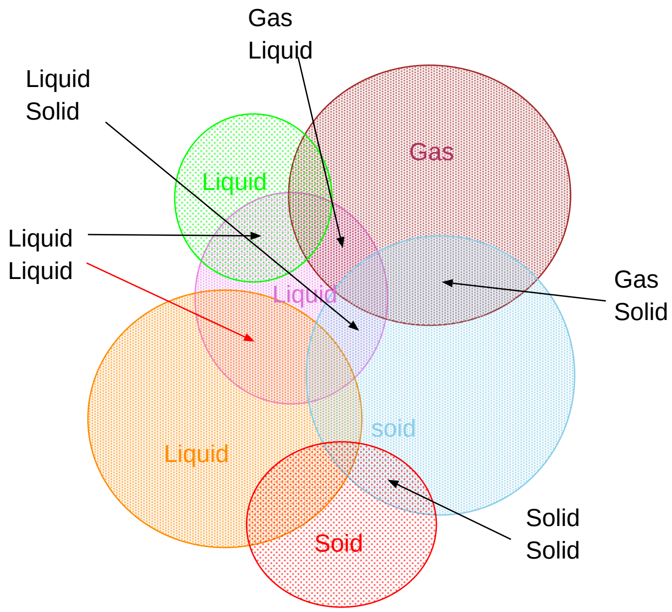


*Fig. -9.1. Different fields of multi phase flow.*

## 9.4 Kind of Multi-Phase Flow

All the flows are a form of multiphase flow. The discussion in the previous chapters is only as approximation when multiphase can be "reduced" into a single phase flow. For example, consider air flow that was discussed and presented earlier as a single phase flow. Air is not a pure material but a mixture of many gases. In fact, many proprieties of air are calculated as if the air is made of well mixed gases of Nitrogen and Oxygen. The results of the calculations of a mixture do not change much if it is assumed that the air flow as stratified flow [4] of many concentration layers (thus, many layers (infinite) of different materials). Practically for many cases, the homogeneous assumption is enough and suitable. However, this assumption will not be appropriate when the air is stratified because of large body forces, or a large acceleration. Adopting this assumption might lead to a larger error. Hence, there are situations when air flow has to be considered as multiphase flow and this effect has to be taken into account.

In our calculation, it is assumed that air is made of only gases. The creation

[3]This author feels that he is in an unique position to influence many in the field of fluid mechanics. This fact is due to the shear number of the downloaded Potto books. The number of the downloads of the book on Fundamental of compressible flow has exceed more than 100,000 in about two and half years. It also provides an opportunity to bring the latest advances in the fields since this author does not need to "sell" the book to a publisher or convince a "committee."

[4]Different concentration of oxygen as a function of the height. While the difference of the concentration between the top to button is insignificant, nonetheless it exists.

of clean room is a proof that air contains small particles. In almost all situations, the cleanness of the air or the fact that air is a mixture is ignored. The engineering accuracy is enough to totally ignore it. Yet, there are situations where cleanness of the air can affect the flow. For example, the cleanness of air can reduce the speed of sound. In the past, the breaks in long trains were activated by reduction of the compressed line (a patent no. 360070 issued to George Westinghouse, Jr., March 29, 1887). In a four (4) miles long train, the breaks would started to work after about 20 seconds in the last wagon. Thus, a 10% change of the speed of sound due to dust particles in air could reduce the stopping time by 2 seconds (50 meter difference in stopping) and can cause an accident.

One way to categorize the multiphase is by the materials flows, For example, the flow of oil and water in one pipe is a multiphase flow. This flow is used by engineers to reduce the cost of moving crude oil through a long pipes system. The "average" viscosity is meaningless since in many cases the water follows around the oil. The water flow is the source of the friction. However, it is more common to categorize the flow by the distinct phases that flow in the tube. Since there are three phases, they can be solid–liquid, solid–gas, liquid–gas and solid–liquid–gas flow. This notion eliminates many other flow categories that can and should be included in multiphase flow. This category should include any distinction of phase/material. There are many more categories, for example, sand and grain (which are "solids") flow with rocks and is referred to solid–solid flow. The category of liquid–gas should be really viewed as the extreme case of liquid-liquid where the density ratio is extremely large. The same can be said for gas–gas flow. For the gas, the density is a strong function of the temperature and pressure. Open Channel flow is, although important, is only an extreme case of liquid-gas flow and is a sub category of the multiphase flow.

The multiphase is an important part of many processes. The multiphase can be found in nature, living bodies (bio–fluids), and industries. Gas–solid can be found in sand storms, and avalanches. The body inhales solid particle with breathing air. Many industries are involved with this flow category such as dust collection, fluidized bed, solid propellant rocket, paint spray, spray casting, plasma and river flow with live creatures (small organisms to large fish) flow of ice berg, mud flow etc. The liquid–solid, in nature can be blood flow, and river flow. This flow also appears in any industrial process that are involved in solidification (for example die casting) and in moving solid particles. Liquid–liquid flow is probably the most common flow in the nature. Flow of air is actually the flow of several light liquids (gases). Many natural phenomenon are multiphase flow, for an example, rain. Many industrial process also include liquid-liquid such as painting, hydraulic with two or more kind of liquids.

## 9.5 Classification of Liquid-Liquid Flow Regimes

The general discussion on liquid–liquid will be provided and the gas–liquid flow will be discussed as a special case. Generally, there are two possibilities for two different materials to flow (it is also correct for solid–liquid and any other combination). The materials can flow in the same direction and it is referred as co–current flow. When the materials flow in the opposite direction, it is referred as counter–current. In general, the co-current is the more common. Additionally, the counter–current flow must have special configurations of long length of flow. Generally, the counter–current flow has a limited length window of possibility in a vertical flow in conduits with the exception of magnetohydrodynamics. The flow regimes are referred to the arrangement of the fluids.

The main difference between the liquid–liquid flow to gas-liquid flow is that gas density is extremely lighter than the liquid density. For example, water and air flow as oppose to water and oil flow. The other characteristic that is different between the gas flow and the liquid flow is the variation of the density. For example, a reduction of the pressure by half will double the gas volumetric flow rate while the change in the liquid is negligible. Thus, the flow of gas–liquid can have several flow regimes in one situation while the flow of liquid–liquid will (probably) have only one flow regime.

## 9.5.1 Co–Current Flow

In Co–Current flow, two liquids can have three main categories: vertical, horizontal, and what ever between them. The vertical configuration has two cases, up or down. It is common to differentiate between the vertical (and near vertical) and horizontal (and near horizontal). There is no exact meaning to the word "near vertical" or "near horizontal" and there is no consensus on the limiting angles (not to mention to have limits as a function with any parameter that determine the limiting angle). The flow in inclined angle (that not covered by the word "near") exhibits flow regimes not much different from the other two. Yet, the limits between the flow regimes are considerably different. This issue of incline flow will not be covered in this chapter.

### 9.5.1.1 Horizontal Flow

The typical regimes for horizontal flow are stratified flow (open channel flow, and non open channel flow), dispersed bubble flow, plug flow, and annular flow. For low velocity (low flow rate) of the two liquids, the heavy liquid flows on the bottom and lighter liquid flows on the top[5] as depicted in Figure 9.2. This kind of flow regime is referred to as horizontal flow. When the flow rate of the lighter liquid is almost zero, the flow is referred to as open channel flow. This definition (open channel flow) continues for small amount of lighter liquid as long as the heavier flow can be calculated as open channel flow (ignoring the lighter liquid). The geometries (even the boundaries) of open channel flow are very diverse. Open channel flow appears in many nature (river) as well in industrial process such as the die casting process where liquid metal is injected into a cylinder (tube) shape. The channel flow will be discussed in a greater detail in Open Channel Flow chapter.

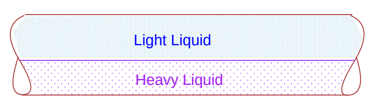


*Fig. -9.2. Stratified flow in horizontal tubes when the liquids flow is very slow.*

[5]With the exception of the extremely smaller diameter where Rayleigh–Taylor instability is an important issue.

As the lighter liquid (or the gas phase) flow rate increases (superficial velocity), the friction between the phases increase. The superficial velocity is referred to as the velocity that any phase will have if the other phase was not exist. This friction is one of the cause for the instability which manifested itself as waves and changing the surface from straight line to a different configuration (see Figure 9.3). The wave shape is created to keep the gas and the liquid velocity equal and at the same time to have shear stress to be balance by surface tension. The configuration of the cross section not only depend on the surface tension, and other physical properties of the fluids but also on the material of the conduit.

As the lighter liquid velocity increases two things can happen (1) wave size increase and (2) the shape of cross section continue to deform. Some referred to this regime as wavy stratified flow but this definition is not accepted by all as a category by itself. In fact, all the two phase flow are categorized by wavy flow which will proven later. There are two paths that can occur on the heavier liquid flow rate. If the heavier flow rate is small, then the wave cannot reach to the crown and the shape is deformed to the point that all the heavier liquid is around the periphery. This kind of flow regime is referred to as annular flow. If the heavier liquid flow rate is larger[6] than the distance, for the wave to reach the conduit crown is smaller. At some point, when the lighter liquid flow increases, the heavier liquid wave reaches to the crown of the pipe. At this stage, the flow pattern is referred to as slug flow or plug flow. Plug flow is characterized by regions of lighter liquid filled with drops of the heavier liquid with Plug (or Slug) of the heavier liquid (with bubble of the lighter liquid). These plugs are separated by large "chunks" that almost fill the entire tube. The plugs are flowing in a succession (see Figure 9.4). The pressure drop of this kind of regime is significantly larger than the stratified flow. The slug flow cannot be assumed to be as homogeneous flow nor it can exhibit some average viscosity. The "average" viscosity depends on the flow and thus making it as insignificant way to do the calculations. Further increase of the lighter liquid flow rate move the flow regime into annular flow. Thus, the possibility to go through slug flow regime depends on if there is enough liquid flow rate.

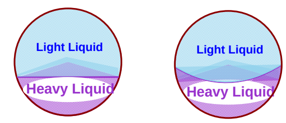


*Fig. -9.3. Kind of Stratified flow in horizontal tubes.*

Choking occurs in compressible flow when the flow rate is above a certain point. All liquids are compressible to some degree. For liquid which the density is a strong and primary function of the pressure, choking occurs relatively closer/sooner. Thus, the flow that starts as a stratified flow will turned into a slug flow or stratified wavy[7] flow after a certain distance depends on the heavy flow rate (if

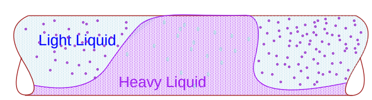


*Fig. -9.4. Plug flow in horizontal tubes when the liquids flow is faster.*

[6]The liquid level is higher.

[7]Well, all the flow is wavy, thus it is arbitrary definition.

this category is accepted). After a certain distance, the flow become annular or the flow will choke. The choking can occur before the annular flow regime is obtained depending on the velocity and compressibility of the lighter liquid. Hence, as in compressible flow, liquid–liquid flow has a maximum combined of the flow rate (both phases). This maximum is known as double choking phenomenon.

The reverse way is referred to the process where the starting point is high flow rate and the flow rate is decreasing. As in many fluid mechanics and magnetic fields, the return path is not move the exact same way. There is even a possibility to return on different flow regime. For example, flow that had slug flow in its path can be returned as stratified wavy flow. This phenomenon is refer to as hysteresis.

Flow that is under small angle from the horizontal will be similar to the horizontal flow. However, there is no consensus how far is the "near" means. Qualitatively, the "near" angle depends on the length of the pipe. The angle decreases with the length of the pipe. Besides the length, other parameters can affect the "near."

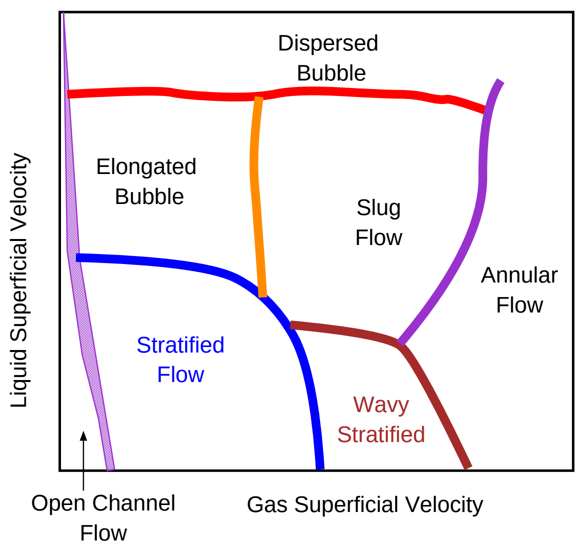


*Fig. -9.5. Modified Mandhane map for flow regime in horizontal tubes.*

The results of the above discussion are depicted in Figure 9.5. As many things in multiphase, this map is only characteristics of the "normal" conditions, e.g. in normal gravitation, weak to strong surface tension effects (air/water in "normal" gravity), etc.

#### 9.5.1.2 Vertical Flow

The vertical flow has two possibilities, with the gravity or against it. In engineering application, the vertical flow against the gravity is more common used. There is a difference between flowing with the gravity and flowing against the gravity. The buoyancy

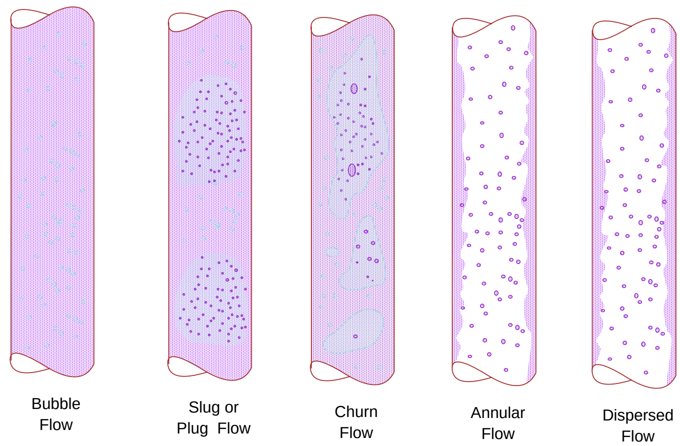


*Fig. -9.6. Gas and liquid in Flow in verstical tube against the gravity.*

is acting in two different directions for these two flow regimes. For the flow against gravity, the lighter liquid has a buoyancy that acts as an "extra force" to move it faster and this effect is opposite for the heavier liquid. The opposite is for the flow with gravity. Thus, there are different flow regimes for these two situations. The main reason that causes the difference is that the heavier liquid is more dominated by gravity (body forces) while the lighter liquid is dominated by the pressure driving forces.

**Flow Against Gravity**

For vertical flow against gravity, the flow cannot start as a stratified flow. The heavier liquid has to occupy almost the entire cross section before it can flow because of the gravity forces. Thus, the flow starts as a bubble flow. The increase of the lighter liquid flow rate will increase the number of bubbles until some bubbles start to collide. When many bubbles collide, they create a large bubble and the flow is referred to as slug flow or plug flow (see Figure 9.6). Notice, the different mechanism in creating the plug flow in horizontal flow compared to the vertical flow.

Further increase of lighter liquid flow rate will increase the slug size as more bubbles collide to create "super slug"; the flow regime is referred as elongated bubble flow. The flow is less stable as more turbulent flow and several "super slug" or churn flow appears in more chaotic way, see Figure 9.6. After additional increase of "super slug" , all these "elongated slug" unite to become an annular flow. Again, it can be noted the difference in the mechanism that create annular flow for vertical and horizontal flow. Any further increase transforms the outer liquid layer into bubbles in the inner liquid. Flow of near vertical against the gravity in two–phase does not deviate from vertical. The choking can occur at any point depends on the fluids and temperature and pressure.

#### 9.5.1.3 Vertical Flow Under Micro Gravity

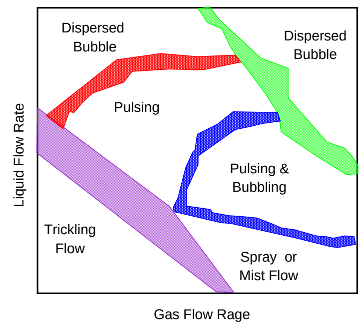


*Fig. -9.7. A dimensional vertical flow map under very low gravity against the gravity.*

The above discussion mostly explained the flow in a vertical configuration when the surface tension can be neglected. In cases where the surface tension is very important. For example, out in space between gas and liquid (large density difference) the situation is different. The flow starts as dispersed bubble (some call it as "gas continuous") because the gas phase occupies most of column. The liquid flows through a trickle or channeled flow that only partially wets part of the tube. The interaction between the phases is minimal and can be considered as the "open channel flow" of the vertical configuration. As the gas flow increases, the liquid becomes more turbulent and some parts enter into the gas phase as drops. When the flow rate of the gas increases further, all the gas phase change into tiny drops of liquid and this kind of regime referred to as mist flow. At a higher rate of liquid flow and a low flow rate of gas, the regime liquid fills the entire void and the gas is in small bubble and this flow referred to as bubbly flow. In the medium range of the flow rate of gas and liquid, there is pulse flow in which liquid is moving in frequent pulses. The common map is based on dimensionless parameters. Here, it is presented in a dimension form to explain the trends (see Figure 9.7). In the literature, Figure 9.7 presented in dimensionless coordinates. The abscissa is a function of combination of Froude ,Reynolds, and Weber numbers. The ordinate is a combination of flow rate ratio and density ratio.

**Flow With The Gravity**

As opposed to the flow against gravity, this flow can starts with stratified flow. A good example for this flow regime is a water fall. The initial part for this flow is more significant. Since the heavy liquid can be supplied from the "wrong" point/side, the initial part has a larger section compared to the flow against the gravity flow. After the flow has settled, the flow continues in a stratified configuration. The transitions between the flow regimes is similar to stratified flow. However, the points where these transitions occur are different from the horizontal flow. While this author is not aware of an actual model, it must be possible to construct a model that connects this configuration with the stratified flow where the transitions will be dependent on the angle of inclinations.

## 9.6 Multi–Phase Flow Variables Definitions

Since the gas–liquid system is a specific case of the liquid–liquid system, both will be united in this discussion. However, for the convenience of the terms "gas and liquid" will be used to signify the lighter and heavier liquid, respectively. The liquid–liquid (also gas–liquid) flow is an extremely complex three–dimensional transient problem since the flow conditions in a pipe may vary along its length, over its cross section, and with time. To simplify the descriptions of the problem and yet to retain the important features of the flow, some variables are defined so that the flow can be described as a one-dimensional flow. This method is the most common and important to analyze two-phase flow pressure drop and other parameters. Perhaps, the only serious missing point in this discussion is the change of the flow along the distance of the tube.

### 9.6.1 Multi–Phase Averaged Variables Definitions

The total mass flow rate through the tube is the sum of the mass flow rates of the two phases

$$\dot{m} = \dot{m}_G + \dot{m}_L \tag{9.1}$$

It is common to define the mass velocity instead of the regular velocity because the "regular" velocity changes along the length of the pipe. The gas mass velocity is

$$G_G = \frac{\dot{m}_G}{A} \tag{9.2}$$

Where $A$ is the entire area of the tube. It has to be noted that this mass velocity does not exist in reality. The liquid mass velocity is

$$G_L = \frac{\dot{m}_L}{A} \tag{9.3}$$

The mass flow of the tube is then

$$G = \frac{\dot{m}}{A} \tag{9.4}$$

It has to be emphasized that this mass velocity is the actual velocity.

The volumetric flow rate is not constant (since the density is not constant) along the flow rate and it is defined as

$$Q_G = \frac{G_G}{\rho_G} = U_{sG} \tag{9.5}$$

and for the liquid

$$Q_L = \frac{G_L}{\rho_L} \tag{9.6}$$

For liquid with very high bulk modulus (almost constant density), the volumetric flow rate can be considered as constant. The total volumetric volume vary along the tube length and is

$$Q = Q_L + Q_G \tag{9.7}$$

Ratio of the gas flow rate to the total flow rate is called the 'quality' or the "dryness fraction" and is given by

$$X = \frac{\dot{m}_G}{\dot{m}} = \frac{G_G}{G} \tag{9.8}$$

In a similar fashion, the value of $(1 - X)$ is referred to as the "wetness fraction." The last two factions remain constant along the tube length as long the gas and liquid masses remain constant. The ratio of the gas flow cross sectional area to the total cross sectional area is referred as the void fraction and defined as

$$\alpha = \frac{A_G}{A} \tag{9.9}$$

This fraction is vary along tube length since the gas density is not constant along the tube length. The liquid fraction or liquid holdup is

$$L_H = 1 - \alpha = \frac{A_L}{A} \tag{9.10}$$

It must be noted that Liquid holdup, $L_H$ is not constant for the same reasons the void fraction is not constant.

The actual velocities depend on the other phase since the actual cross section the phase flows is dependent on the other phase. Thus, a superficial velocity is commonly defined in which if only one phase is using the entire tube. The gas superficial velocity is therefore defined as

$$U_{sG} = \frac{G_G}{\rho_G} = \frac{X\,\dot{m}}{\rho_G\,A} = Q_G \tag{9.11}$$

The liquid superficial velocity is

$$U_{sL} = \frac{G_L}{\rho_L} = \frac{(1 - X)\,\dot{m}}{\rho_L\,A} = Q_L \tag{9.12}$$

Since $U_{sL} = Q_L$ and similarly for the gas then

$$U_m = U_{sG} + U_{sL} \tag{9.13}$$

Where $U_m$ is the averaged velocity. It can be noticed that $U_m$ is not constant along the tube.

The average superficial velocity of the gas and liquid are different. Thus, the ratio of these velocities is referred to as the slip velocity and is defined as the following

$$SLP = \frac{U_G}{U_L} \tag{9.14}$$

Slip ratio is usually greater than unity. Also, it can be noted that the slip velocity is not constant along the tube.

For the same velocity of phases ($SLP = 1$), the mixture density is defined as

$$\rho_m = \alpha\,\rho_G + (1-\alpha)\,\rho_L \tag{9.15}$$

This density represents the density taken at the "frozen" cross section (assume the volume is the cross section times infinitesimal thickness of $dx$).

The average density of the material flowing in the tube can be evaluated by looking at the definition of density. The density of any material is defined as $\rho = m/V$ and thus, for the flowing material it is

$$\rho = \frac{\dot{m}}{Q} \tag{9.16}$$

Where $Q$ is the volumetric flow rate. Substituting equations (9.1) and (9.7) into equation (9.16) results in

$$\rho_{average} = \frac{\overbrace{X\,\dot{m}}^{\dot{m}_G} + \overbrace{(1-X)\,\dot{m}}^{\dot{m}_L}}{Q_G + Q_L} = \frac{X\,\dot{m} + (1-X)\,\dot{m}}{\underbrace{\frac{X\,\dot{m}}{\rho_G}}_{Q_G} + \underbrace{\frac{(1-X)\,\dot{m}}{\rho_L}}_{Q_L}} \tag{9.17}$$

Equation (9.17) can be simplified by canceling the $\dot{m}$ and noticing the $(1-X)+X = 1$ to become

$$\rho_{average} = \frac{1}{\frac{X}{\rho_G} + \frac{(1-X)}{\rho_L}} \tag{9.18}$$

The average specific volume of the flow is then

$$v_{average} = \frac{1}{\rho_{average}} = \frac{X}{\rho_G} + \frac{(1-X)}{\rho_L} = X\,v_G + (1-X)\,v_L \tag{9.19}$$

The relationship between $X$ and $\alpha$ is

$$X = \frac{\dot{m}_G}{\dot{m}_G + \dot{m}_L} = \frac{\rho_G\,U_G\,\overbrace{A\,\alpha}^{A_G}}{\rho_L U_L\,\underbrace{A(1-\alpha)}_{A_L} + \rho_G\,U_G\,A\,\alpha} = \frac{\rho_G\,U_G\,\alpha}{\rho_L U_L (1-\alpha) + \rho_G\,U_G\,\alpha} \tag{9.20}$$

If the slip is one $SLP = 1$, thus equation (9.20) becomes

$$X = \frac{\rho_G\ \alpha}{\rho_L(1-\alpha) + \rho_G\,\alpha} \tag{9.21}$$

## 9.7 Homogeneous Models

Before discussing the homogeneous models, it is worthwhile to appreciate the complexity of the flow. For the construction of fluid basic equations, it was assumed that the flow is continuous. Now, this assumption has to be broken, and the flow is continuous only in many chunks (small segments). Furthermore, these segments are not defined but results of the conditions imposed on the flow. In fact, the different flow regimes are examples of typical configuration of segments of continuous flow. Initially, it was assumed that the different flow regimes can be neglected at least for the pressure loss (not correct for the heat transfer). The single phase was studied earlier in this book and there is a considerable amount of information about it. Thus, the simplest is to used it for approximation.

The average velocity (see also equation (9.13)) is

$$U_m = \frac{Q_L + Q_G}{A} = U_{sL} + U_{sG} = U_m \tag{9.22}$$

It can be noted that the continuity equation is satisfied as

$$\boxed{\dot{m} = \rho_m \, U_m \, A} \tag{9.23}$$

Example 9.1:
*Under what conditions equation* (9.23) *is correct?*

SOLUTION

Under construction

End Solution

The governing momentum equation can be approximated as

$$\dot{m} \frac{dU_m}{dx} = -A \frac{dP}{dx} - S \, \tau_w - A \, \rho_m \, g \, \sin\theta \tag{9.24}$$

or modifying equation (9.24) as

$$\boxed{-\frac{dP}{dx} = -\frac{S}{A} \tau_w - \frac{\dot{m}}{A} \frac{dU_m}{dx} + \rho_m \, g \, \sin\theta} \tag{9.25}$$

The energy equation can be approximated as

$$\boxed{\frac{dq}{dx} - \frac{dw}{dx} = \dot{m} \frac{d}{dx} \left( h_m + \frac{{U_m}^2}{2} + g \, x \, \sin\theta \right)} \tag{9.26}$$

### 9.7.1 Pressure Loss Components

In a tube flowing upward in incline angle $\theta$, the pressure loss is affected by friction loss, acceleration, and body force(gravitation). These losses are non-linear and depend on each other. For example, the gravitation pressure loss reduce the pressure and thus the density must change and hence, acceleration must occur. However, for small distances ($dx$) and some situations, this dependency can be neglected. In that case, from equation (9.25), the total pressure loss can be written as

$$\frac{dP}{dx} = \overbrace{\left.\frac{dP}{dx}\right|_{f}}^{friction} + \overbrace{\left.\frac{dP}{dx}\right|_{a}}^{acceleration} + \overbrace{\left.\frac{dP}{dx}\right|_{g}}^{gravity} \tag{9.27}$$

Every part of the total pressure loss will be discussed in the following section.

#### 9.7.1.1 Friction Pressure Loss

The frictional pressure loss for a conduit can be calculated as

$$-\left.\frac{dP}{dx}\right|_{f} = \frac{S}{A}\tau_w \tag{9.28}$$

Where $S$ is the perimeter of the fluid. For calculating the frictional pressure loss in the pipe is

$$-\left.\frac{dP}{dx}\right|_{f} = \frac{4\,\tau_w}{D} \tag{9.29}$$

The wall shear stress can be estimated by

$$\tau_w = f\,\frac{\rho_m\,{U_m}^2}{2} \tag{9.30}$$

The friction factor is measured for a single phase flow where the average velocity is directly related to the wall shear stress. There is not available experimental data for the relationship of the averaged velocity of the two (or more) phases and wall shear stress. In fact, this friction factor was not measured for the "averaged" viscosity of the two phase flow. Yet, since there isn't anything better, the experimental data that was developed and measured for single flow is used.

The friction factor is obtained by using the correlation

$$f = C\left(\frac{\rho_m\,U_m\,D}{\mu_m}\right)^{-n} \tag{9.31}$$

Where $C$ and $n$ are constants which depend on the flow regimes (turbulent or laminar flow). For laminar flow $C = 16$ and $n = 1$. For turbulent flow $C = 0.079$ and $n = 0.25$.

There are several suggestions for the average viscosity. For example, Duckler suggest the following

$$\mu_m = \frac{\mu_G\, Q_G}{Q_G + Q_L} + \frac{\mu_L\, Q_L}{Q_G + Q_L} \tag{9.32}$$

Duckler linear formula does not provide always good approximation and Cichilli suggest similar to equation (9.18) average viscosity as

$$\mu_{average} = \frac{1}{\frac{X}{\mu_G} + \frac{(1-X)}{\mu_L}} \tag{9.33}$$

Or simply make the average viscosity depends on the mass fraction as

$$\mu_m = X\,\mu_G + (1-X)\,\mu_L \tag{9.34}$$

Using this formula, the friction loss can be estimated.

#### 9.7.1.2 Acceleration Pressure Loss

The acceleration pressure loss can be estimated by

$$-\left.\frac{dP}{dx}\right|_a = \dot{m}\frac{dU_m}{dx} \tag{9.35}$$

The acceleration pressure loss (can be positive or negative) results from change of density and the change of cross section. Equation (9.35) can be written as

$$-\left.\frac{dP}{dx}\right|_a = \dot{m}\,\frac{d}{dx}\left(\frac{\dot{m}}{A\,\rho_m}\right) \tag{9.36}$$

Or in an explicit way equation (9.36) becomes

$$-\left.\frac{dP}{dx}\right|_a = \dot{m}^2 \left[ \overbrace{\frac{1}{A}\frac{d}{dx}\left(\frac{1}{\rho_m}\right)}^{\text{pressure loss due to density change}} + \overbrace{\frac{1}{\rho_m\, A^2}\frac{dA}{dx}}^{\text{pressure loss due to area change}} \right] \tag{9.37}$$

There are several special cases. The first case where the cross section is constant, $dA/\,dx = 0$. In second case is where the mass flow rates of gas and liquid is constant in which the derivative of $X$ is zero, $dX/\,dx = 0$. The third special case is for constant density of one phase only, $d\rho_L/\,dx = 0$. For the last point, the private case is where densities are constant for both phases.

#### 9.7.1.3 Gravity Pressure Loss

Gravity was discussed in Chapter 4 and is

$$\left.\frac{dP}{dx}\right|_{g} = g\,\rho_m\,\sin\theta \tag{9.38}$$

The density change during the flow can be represented as a function of density. The density in equation (9.38) is the density without the "movement" (the "static" density).

#### 9.7.1.4 Total Pressure Loss

The total pressure between two points, ($a$ and $b$) can be calculated with integration as

$$\Delta P_{ab} = \int_a^b \frac{dP}{dx} dx \tag{9.39}$$

and therefore

$$\Delta P_{ab} = \overbrace{\Delta P_{ab\,f}}^{friction} + \overbrace{\Delta P_{ab\,a}}^{acceleration} + \overbrace{\Delta P_{ab\,g}}^{gravity} \tag{9.40}$$

### 9.7.2 Lockhart Martinelli Model

The second method is by assumption that every phase flow separately One such popular model by Lockhart and Martinelli[8]. Lockhart and Martinelli built model based on the assumption that the separated pressure loss are independent from each other. Lockhart Martinelli parameters are defined as the ratio of the pressure loss of two phases and pressure of a single phase. Thus, there are two parameters as shown below.

$$\phi_G = \left.\sqrt{\left.\frac{dP}{dx}\right|_{TP} \Big/ \left.\frac{dP}{dx}\right|_{SG}}\,\right|_f \tag{9.41}$$

Where the $TP$ denotes the two phases and $SG$ denotes the pressure loss for the single gas phase. Equivalent definition for the liquid side is

$$\phi_L = \left.\sqrt{\left.\frac{dP}{dx}\right|_{TP} \Big/ \left.\frac{dP}{dx}\right|_{SL}}\,\right|_f \tag{9.42}$$

Where the $SL$ denotes the pressure loss for the single liquid phase.

[8] This method was considered a military secret, private communication with Y., Taitle

The ratio of the pressure loss for a single liquid phase and the pressure loss for a single gas phase is

$$\Xi = \sqrt{\left.\frac{dP}{dx}\right|_{SL} \Big/ \left.\frac{dP}{dx}\right|_{SG}}\ \Bigg|_f \tag{9.43}$$

where $\Xi$ is Martinelli parameter.

It is assumed that the pressure loss for both phases are equal.

$$\left.\frac{dP}{dx}\right|_{SG} = \left.\frac{dP}{dx}\right|_{SL} \tag{9.44}$$

The pressure loss for the liquid phase is

$$\left.\frac{dP}{dx}\right|_{L} = \frac{2\, f_L\, {U_L}^2\, \rho_l}{D_L} \tag{9.45}$$

For the gas phase, the pressure loss is

$$\left.\frac{dP}{dx}\right|_{G} = \frac{2\, f_G\, {U_G}^2\, \rho_l}{D_G} \tag{9.46}$$

Simplified model is when there is no interaction between the two phases.

To insert the Diagram.

## 9.8 Solid–Liquid Flow

Solid–liquid system is simpler to analyze than the liquid-liquid system. In solid–liquid, the effect of the surface tension are very minimal and can be ignored. Thus, in this discussion, it is assumed that the surface tension is insignificant compared to the gravity forces. The word "solid" is not really mean solid but a combination of many solid particles. Different combination of solid particle creates different "liquid." Therefor,there will be a discussion about different particle size and different geometry (round, cubic, etc). The uniformity is categorizing the particle sizes, distribution, and geometry. For example, analysis of small coal particles in water is different from large coal particles in water.

The density of the solid can be above or below the liquid. Consider the case where the solid is heavier than the liquid phase. It is also assumed that the "liquids" density does not change significantly and it is far from the choking point. In that case there are four possibilities for vertical flow:

1. The flow with the gravity and lighter density solid particles.
2. The flow with the gravity and heavier density solid particles.
3. The flow against the gravity and lighter density solid particles.

4. The flow against the gravity and heavier density solid particles.

All these possibilities are different. However, there are two sets of similar characteristics, possibility, 1 and 4 and the second set is 2 and 3. The first set is similar because the solid particles are moving faster than the liquid velocity and vice versa for the second set (slower than the liquid). The discussion here is about the last case (4) because very little is known about the other cases.

### 9.8.1 Solid Particles with Heavier Density $\rho_S > \rho_L$

Solid–liquid flow has several combination flow regimes.

When the liquid velocity is very small, the liquid cannot carry the solid particles because there is not enough resistance to lift up the solid particles. A particle in a middle of the vertical liquid flow experience several forces. The force balance of spherical particle in field viscous fluid (creeping flow) is

$$\overbrace{\frac{\pi\, D^3\, g\,(\rho_S - \rho_L)}{6}}^{\substack{\text{gravity and buoyancy}\\ \text{forces}}} = \overbrace{\frac{C_{D\infty}\, \pi\, D^2\, \rho_L\, {U_L}^2}{8}}^{\text{drag forces}} \tag{9.47}$$

Where $C_{D\infty}$ is the drag coefficient and is a function of Reynolds number, $Re$, and $D$ is the equivalent radius of the particles. The Reynolds number defined as

$$Re = \frac{U_L\, D\, \rho_L}{\mu_L} \tag{9.48}$$

Inserting equating (9.48) into equation (9.47) become

$$\overbrace{f(Re)}^{C_{D\infty}(U_L)}\ {U_L}^2 = \frac{4\, D\, g\,(\rho_S - \rho_L)}{3\, \rho_L} \tag{9.49}$$

Equation (9.49) relates the liquid velocity that needed to maintain the particle "floating" to the liquid and particles properties. The drag coefficient, $C_{D\infty}$ is complicated function of the Reynolds number. However, it can be approximated for several regimes. The first regime is for $Re < 1$ where Stokes' Law can be approximated as

$$C_{D\infty} = \frac{24}{Re} \tag{9.50}$$

In transitional region $1 < Re < 1000$

$$C_{D\infty} = \frac{24}{Re}\left(1 + \frac{1}{6}\, Re^{2/3}\right) \tag{9.51}$$

For larger Reynolds numbers, the Newton's Law region, $C_{D\infty}$, is nearly constant as

$$C_{D\infty} = 0.44 \tag{9.52}$$

In most cases of solid-liquid system, the Reynolds number is in the second range[9]. For the first region, the velocity is small to lift the particle unless the density difference is very small (that very small force can lift the particles). In very large range (especially for gas) the choking might be approached. Thus, in many cases the middle region is applicable.

So far the discussion was about single particle. When there are more than one particle in the cross section, then the actual velocity that every particle experience depends on the void fraction. The simplest assumption that the change of the cross section of the fluid create a parameter that multiply the single particle as

$$C_{D\infty}\big|_{\alpha} = C_{D\infty}\, f(\alpha) \tag{9.53}$$

When the subscript $\alpha$ is indicating the void, the function $f(\alpha)$ is not a linear function. In the literature there are many functions for various conditions.

Minimum velocity is the velocity when the particle is "floating". If the velocity is larger, the particle will drift with the liquid. When the velocity is lower, the particle will sink into the liquid. When the velocity of liquid is higher than the minimum velocity many particles will be floating. It has to remember that not all the particle are uniform in size or shape. Consequently, the minimum velocity is a range of velocity rather than a sharp transition point.

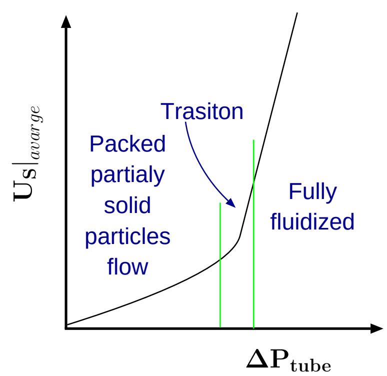


*Fig. -9.8. The terminal velocity that left the solid particles.*

As the solid particles are not pushed by a pump but moved by the forces the fluid applies to them. Thus, the only velocity that can be applied is the fluid velocity. Yet, the solid particles can be supplied at different rate. Thus, the discussion will be focus on the fluid velocity. For small gas/liquid velocity, the particles are what some call fixed fluidized bed. Increasing the fluid velocity beyond a minimum will move the particles and it is referred to as mix fluidized bed. Additional increase of the fluid velocity will move all the particles and this is referred to as fully fluidized bed. For the case of liquid, further increase will create a slug flow. This slug flow is when slug shape (domes) are almost empty of the solid particle. For the case of gas, additional increase create "tunnels" of empty almost from solid particles. Additional increase in the fluid velocity causes large turbulence and the ordinary domes are replaced by churn type flow or large bubbles that are almost empty of the solid particles. Further increase of the fluid flow increases the empty spots to the whole flow. In that case, the sparse solid particles are dispersed all over. This regimes is referred to as Pneumatic conveying (see Figure 9.9).

[9]It be wonderful if flow was in the last range? The critical velocity could be found immediately.

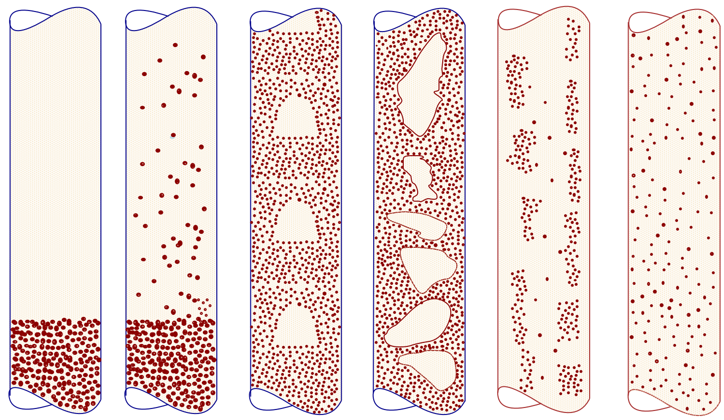


*Fig. -9.9. The flow patterns in solid-liquid flow.*

One of the main difference between the liquid and gas flow in this category is the speed of sound. In the gas phase, the speed of sound is reduced dramatically with increase of the solid particles concentration (further reading Fundamentals of Compressible Flow" chapter on Fanno Flow by this author is recommended). Thus, the velocity of gas is limited when reaching the Mach somewhere between $1/\sqrt{k}$ and $1$ since the gas will be choked (neglecting the double choking phenomenon). Hence, the length of conduit is very limited. The speed of sound of the liquid does not change much. Hence, this limitation does not (effectively) exist for most cases of solid–liquid flow.

### 9.8.2 Solid With Lighter Density $\rho_S < \rho$ and With Gravity

This situation is minimal and very few cases exist. However, it must be pointed out that even in solid–gas, the fluid density can be higher than the solid (especially with micro gravity). There was very little investigations and known about the solid–liquid flowing down (with the gravity). Furthermore, there is very little knowledge about the solid–liquid when the solid density is smaller than the liquid density. There is no known flow map for this kind of flow that this author is aware of.

Nevertheless, several conclusions and/or expectations can be drawn. The issue of minimum terminal velocity is not exist and therefor there is no fixed or mixed fluidized bed. The flow is fully fluidized for any liquid flow rate. The flow can have slug flow but more likely will be in fast Fluidization regime. The forces that act on the spherical particle are the buoyancy force and drag force. The buoyancy is accelerating the particle and drag force are reducing the speed as

$$\frac{\pi\, D^3\, g(\rho_S - \rho_L)}{6} = \frac{C_{D\infty}\, \pi\, D^2 \rho_L\, (U_S - U_L)^2}{8} \tag{9.54}$$

From equation 9.54, it can observed that increase of the liquid velocity will increase the solid particle velocity at the same amount. Thus, for large velocity of the fluid it can be observed that $U_L/U_S \rightarrow 1$. However, for a small fluid velocity the velocity ratio is very large, $U_L/U_S \rightarrow 0$. The affective body force "seems" by the particles can be in some cases larger than the gravity. The flow regimes will be similar but the transition will be in different points.

The solid–liquid horizontal flow has some similarity to horizontal gas–liquid flow. Initially the solid particles will be carried by the liquid to the top. When the liquid velocity increase and became turbulent, some of the particles enter into the liquid core. Further increase of the liquid velocity appear as somewhat similar to slug flow. However, this author have not seen any evidence that show the annular flow does not appear in solid–liquid flow.

## 9.9 Counter–Current Flow

This discussion will be only on liquid–liquid systems (which also includes liquid-gas systems). This kind of flow is probably the most common to be realized by the masses. For example, opening a can of milk or juice. Typically if only one hole is opened on the top of the can, the liquid will flow in pulse regime. Most people know that two holes are needed to empty the can easily and continuously. Otherwise, the flow will be in a pulse regime.

In most cases, the possibility to have counter–current flow is limited to having short length of tubes. In only certain configurations of the infinite long pipes the counter–current flow can exist. In that case, the pressure difference and gravity (body forces) dominates the flow. The inertia components of the flow, for long tubes, cannot compensate for the pressure gradient. In short tube, the pressure difference in one phase can be positive while the pressure difference in the other phase can be negative. The pressure difference in the interface must be finite. Hence, the counter–current flow can have opposite pressure gradient for short conduit. But in most cases, the heavy phase (liquid) is pushed by the gravity and lighter phase (gas) is driven by the pressure difference.

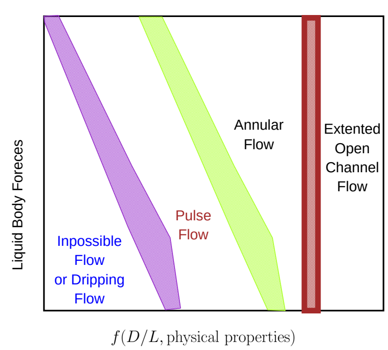


*Fig. -9.10. Counter–flow in vertical tubes map.*

The counter-current flow occurs, for example, when cavity is filled or emptied with a liquid. The two phase regimes "occurs" mainly in entrance to the cavity. For example,

*Fig. -9.11. Counter–current flow in a can (the left figure) has only one hole thus pulse flow and a flow with two holes (right picture).*

Figure 9.11 depicts emptying of can filled with liquid. The air is "attempting" to enter the cavity to fill the vacuum created thus forcing pulse flow. If there are two holes, in some cases, liquid flows through one hole and the air through the second hole and the flow will be continuous. It also can be noticed that if there is one hole (orifice) and a long and narrow tube, the liquid will stay in the cavity (neglecting other phenomena such as dripping flow.).

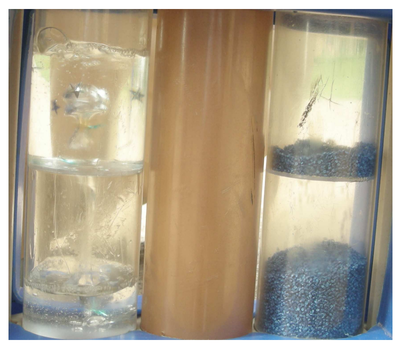

*Fig. -9.12. Picture of Counter-current flow in liquid–gas and solid–gas configurations. The container is made of two compartments. The upper compartment is filled with the heavy phase (liquid, water solution, or small wood particles) by rotating the container. Even though the solid–gas ratio is smaller, it can be noticed that the solid–gas is faster than the liquid–gas flow.*

There are three flow regimes[10] that have been observed. The first flow pattern is pulse flow regime. In this flow regime, the phases flow turns into different direction (see Figure 9.12). The name pulse flow is used to signify that the flow is flowing in pulses that occurs in a certain frequency. This is opposed to counter–current solid–gas flow when almost no pulse was observed. Initially, due to the gravity, the heavy liquid is leaving the can. Then the pressure in the can is reduced compared to the outside and some lighter liquid (gas)entered into the can. Then, the pressure in the can increase,

[10] Caution! this statement should be considered as "so far found". There must be other flow regimes that were not observed or defined. For example, elongated pulse flow was observed but measured. This field hasn't been well explored. There are more things to be examined and to be studied.

and some heavy liquid will starts to flow. This process continue until almost the liquid is evacuated (some liquid stay due the surface tension). In many situations, the volume flow rate of the two phase is almost equal. The duration the cycle depends on several factors. The cycle duration can be replaced by frequency. The analysis of the frequency is much more complex issue and will not be dealt here.

**Annular Flow in Counter–current flow**

The other flow regime is annular flow in which the heavier phase is on the periphery of the conduit (In the literature, there are someone who claims that heavy liquid will be inside). The analysis is provided, but somehow it contradicts with the experimental evidence. Probably, one or more of the assumptions that the analysis based is erroneous). In very small diameters of tubes the counter–current flow is not possible because of the surface tension (see section 4.7). The ratio of the diameter to the length with some combinations of the physical properties (surface tension etc) determines the point where the counter flow can start. At this point, the pulsing flow will start and larger diameter will increase the flow and turn the flow into annular flow. Additional increase of the diameter will change the flow regime into extended open channel flow. Extended open channel flow retains the characteristic of open channel that the lighter liquid (almost) does not effect the heavier liquid flow. Example of such flow in the nature is water falls in which water flows down and air (wind) flows up.

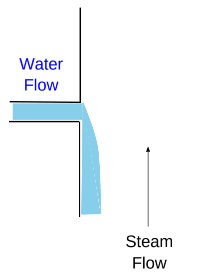


*Fig. -9.13. Flood in vertical pipe.*

The driving force is the second parameter which effects the flow existence. When the driving (body) force is very small, no counter–current flow is possible. Consider the can in zero gravity field, no counter–current flow possible. However, if the can was on the sun (ignoring the heat transfer issue), the flow regime in the can moves from pulse to annular flow. Further increase of the body force will move the flow to be in the extended "open channel flow."

In the vertical co–current flow there are two possibilities, flow with gravity or against it. As opposed to the co–current flow, the counter–current flow has no possibility for these two cases. The heavy liquid will flow with the body forces (gravity). Thus it should be considered as non existent flow.

### 9.9.1 Horizontal Counter–Current Flow

Up to this point, the discussion was focused on the vertical tubes. In horizontal tubes, there is an additional flow regime which is stratified . Horizontal flow is different from vertical flow from the stability issues. A heavier liquid layer can flow above a lighter liquid. This situation is unstable for large diameter but as in static (see section (4.7) page 137) it can be considered stable for small diameters. A flow in a very narrow tube with heavy fluid above the lighter fluid should be considered as a separate issue.

When the flow rate of both fluids is very small, the flow will be stratified counter–current flow. The flow will change to pulse flow when the heavy liquid flow rate increases. Further increase of the flow will result in a single phase flow regime. Thus, closing the window of this kind of flow. Thus, this increase terminates the two phase flow possibility. The flow map of the horizontal flow is different from the vertical flow and is shown in Figure 9.14. A flow in an angle of inclination is closer to vertical flow unless the angle of inclination is very small. The stratified counter flow has a lower pressure loss (for the liquid side). The change to pulse flow increases the pressure loss dramatically.

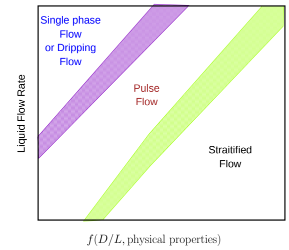


*Fig. -9.14. A flow map to explain the horizontal counter–current flow.*

### 9.9.2 Flooding and Reversal Flow

The limits of one kind the counter–current flow regimes, that is stratified flow are discussed here. This problem appears in nuclear engineering (or boiler engineering) where there is a need to make sure that liquid (water) inserted into the pipe reaching the heating zone. When there is no water (in liquid phase), the fire could melt or damage the boiler. In some situations, the fire can be too large or/and the water supply failed below a critical value the water turn into steam. The steam will flow in the opposite direction. To analyze this situation consider a two dimensional conduit with a liquid inserted in the left side as depicted in Figure 9.13. The liquid velocity at very low gas velocity is constant but not uniform. Further increase of the gas velocity will reduce the average liquid velocity. Additional increase of the gas velocity will bring it to a point where the liquid will flow in a reverse direction and/or disappear (dried out).

A simplified model for this situation is for a two dimensional configuration where the liquid is flowing down and the gas is flowing up as shown in Figure 9.15. It is assumed that both fluids are flowing in a laminar regime and steady state. Additionally, it is assumed that the entrance effects can be neglected. The liquid flow rate, $Q_L$, is unknown. However, the pressure difference in the ($x$ direction) is known and equal to zero. The boundary conditions for the liquid is that velocity at the wall is zero and the velocity at the interface is the same for both phases $U_G = U_L$ or $\tau_i|_G = \tau_i|_L$. As it will be shown later, both conditions cannot coexist. The model can be improved by considering turbulence, mass transfer, wavy interface, etc[11].

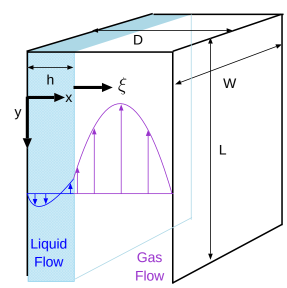


*Fig. -9.15. A diagram to explain the flood in a two dimension geometry.*

[11]The circular configuration is under construction and will be appeared as a separated article momentarily.

This model is presented to exhibits the trends and the special features of counter-current flow. Assuming the pressure difference in the flow direction for the gas is constant and uniform. It is assumed that the last assumption does not contribute or change significantly the results. The underline rational for this assumption is that gas density does not change significantly for short pipes (for more information look for the book "Fundamentals of Compressible Flow" in Potto book series in the Fanno flow chapter.).

The liquid film thickness is unknown and can be expressed as a function of the above boundary conditions. Thus, the liquid flow rate is a function of the boundary conditions. On the liquid side, the gravitational force has to be balanced by the shear forces as

$$\frac{d\tau_{xy}}{dx} = \rho_L\, g \tag{9.55}$$

The integration of equation (9.55) results in

$$\tau_{xy} = \rho_L\, g\, x + C_1 \tag{9.56}$$

The integration constant, $C_1$, can be found from the boundary condition where $\tau_{xy}(x = h) = \tau_i$. Hence,

$$\tau_i = \rho_L\, g\, h + C_1 \tag{9.57}$$

The integration constant is then $C_i = \tau_i - \rho_L\, g\, h$ which leads to

$$\tau_{xy} = \rho_L\, g\,(x - h) + \tau_i \tag{9.58}$$

Substituting the newtonian fluid relationship into equation (9.58) to obtained

$$\mu_L \frac{dU_y}{dx} = \rho_L\, g\,(x - h) + \tau_i \tag{9.59}$$

or in a simplified form as

$$\frac{dU_y}{dx} = \frac{\rho_L\, g\,(x - h)}{\mu_L} + \frac{\tau_i}{\mu_L} \tag{9.60}$$

Equation (9.60) can be integrate to yield

$$U_y = \frac{\rho_L\, g}{\mu_L}\left(\frac{x^2}{2} - h\, x\right) + \frac{\tau_i\, x}{\mu_L} + C_2 \tag{9.61}$$

The liquid velocity at the wall, [$U(x = 0) = 0$], is zero and the integration coefficient can be found to be

$$C_2 = 0 \tag{9.62}$$

The liquid velocity profile is then

$$\boxed{U_y = \frac{\rho_L\, g}{\mu_L}\left(\frac{x^2}{2} - h\, x\right) + \frac{\tau_i\, x}{\mu_L}} \tag{9.63}$$

The velocity at the liquid–gas interface is

$$U_y(x=h) = \frac{\tau_i\,h}{\mu_L} - \frac{\rho_L\,g\,h^2}{2\,\mu_L} \tag{9.64}$$

The velocity can vanish (zero) inside the film in another point which can be obtained from

$$0 = \frac{\rho_L\,g}{\mu_L}\left(\frac{x^2}{2} - h\,x\right) + \frac{\tau_i\,x}{\mu_L} \tag{9.65}$$

The solution for equation (9.65) is

$$x|_{@U_L=0} = 2\,h - \frac{2\,\tau_i}{\mu_L\,g\,\rho_L} \tag{9.66}$$

The maximum $x$ value is limited by the liquid film thickness, $h$. The minimum shear stress that start to create reversible velocity is obtained when $x = h$ which is

$$0 = \frac{\rho_L\,g}{\mu_L}\left(\frac{h^2}{2} - h\,h\right) + \frac{\tau_i\,h}{\mu_L} \tag{9.67}$$

$$\hookrightarrow \tau_{i0} = \frac{h\,g\,\rho_L}{2}$$

If the shear stress is below this critical shear stress $\tau_{i0}$ then no part of the liquid will have a reversed velocity. The notation of $\tau_{i0}$ denotes the special value at which a starting shear stress value is obtained to have reversed flow. The point where the liquid flow rate is zero is important and it is referred to as initial flashing point.

The flow rate can be calculated by integrating the velocity across the entire liquid thickness of the film.

$$\frac{Q}{w} = \int_0^h U_y dx = \int_0^h \left[\frac{\rho_L\,g}{\mu_L}\left(\frac{x^2}{2} - h\,x\right) + \frac{\tau_i\,x}{\mu_L}\right] dx \tag{9.68}$$

Where $w$ is the thickness of the conduit (see Figure 9.15). Integration equation (9.68) results in

$$\frac{Q}{w} = \frac{h^2\,(3\,\tau_i - 2\,g\,h\,\rho_L)}{6\,\mu_L} \tag{9.69}$$

It is interesting to find the point where the liquid mass flow rate is zero. This point can be obtained when equation (9.69) is equated to zero. There are three solutions for equation (9.69). The first two solutions are identical in which the film height is $h = 0$ and the liquid flow rate is zero. But, also, the flow rate is zero when $3\,\tau_i = 2\,g\,h\,\rho_L$. This request is identical to the demand in which

$$\boxed{\tau_{i_{\text{critical}}} = \frac{2\,g\,h\,\rho_L}{3}} \tag{9.70}$$

This critical shear stress, for a given film thickness, reduces the flow rate to zero or effectively "drying" the liquid (which is different then equation (9.67)).

For this shear stress, the critical upward interface velocity is

$$\left. U_{critical} \right|_{interface} = \overbrace{\frac{1}{6}}^{\left(\frac{2}{3}-\frac{1}{2}\right)} \left( \frac{\rho_L \, g \, h^2}{\mu_L} \right) \tag{9.71}$$

The wall shear stress is the last thing that will be done on the liquid side. The wall shear stress is

$$\left. \tau_L \right|_{@wall} = \mu_L \left. \frac{dU}{dx} \right|_{x=0} = \mu_L \left( \frac{\rho_L \, g}{\mu_L} \left( \cancelto{0}{2\,x} - h \right) + \overbrace{\frac{2\, g\, h\, \rho_L}{3}}^{\tau_i} \frac{1}{\mu_L} \right)_{x=0} \tag{9.72}$$

Simplifying equation (9.72)[12] becomes (notice the change of the sign accounting for the direction)

$$\left. \tau_L \right|_{@wall} = \frac{g\, h\, \rho_L}{3} \tag{9.73}$$

Again, the gas is assumed to be in a laminar flow as well. The shear stress on gas side is balanced by the pressure gradient in the $y$ direction. The momentum balance on element in the gas side is

$$\frac{d\tau_{xy\,G}}{dx} = \frac{dP}{dy} \tag{9.74}$$

The pressure gradient is a function of the gas compressibility. For simplicity, it is assumed that pressure gradient is linear. This assumption means or implies that the gas is incompressible flow. If the gas was compressible with an ideal gas equation of state then the pressure gradient is logarithmic. Here, for simplicity reasons, the linear equation is used. In reality the logarithmic equation should be used ( a discussion can be found in "Fundamentals of Compressible Flow" a Potto project book). Thus, equation (9.74) can be rewritten as

$$\frac{d\tau_{xy\,G}}{dx} = \frac{\Delta P}{\Delta y} = \frac{\Delta P}{L} \tag{9.75}$$

Where $\Delta y = L$ is the entire length of the flow and $\Delta P$ is the pressure difference of the entire length. Utilizing the Newtonian relationship, the differential equation is

$$\frac{d^2 U_G}{dx^2} = \frac{\Delta P}{\mu_G \, L} \tag{9.76}$$

[12]Also noticing that equation (9.70) has to be equal $g\,h\,\rho_L$ to support the weight of the liquid.

Equation (9.76) can be integrated twice to yield

$$U_G = \frac{\Delta P}{\mu_G\, L}\, x^2 + C_1\, x + C_2 \tag{9.77}$$

This velocity profile must satisfy zero velocity at the right wall. The velocity at the interface is the same as the liquid phase velocity or the shear stress are equal. Mathematically these boundary conditions are

$$U_G(x = D) = 0 \tag{9.78}$$

and

$$\begin{array}{lcc} U_G(x = h) = U_L(x = h) & (a) & or \\ \tau_G(x = h) = \tau_L(x = h) & (b) & \end{array} \tag{9.79}$$

Applying B.C. (9.78) into equation (9.77) results in

$$\begin{aligned} U_G = 0 &= \frac{\Delta P}{\mu_G\, L}\, D^2 + C_1\, D + C_2 \\ \hookrightarrow C_2 &= -\frac{\Delta P}{\mu_G\, L}\, D^2 + C_1\, D \end{aligned} \tag{9.80}$$

Which leads to

$$U_G = \frac{\Delta P}{\mu_G\, L}\, \left(x^2 - D^2\right) + C_1\, (x - D) \tag{9.81}$$

At the other boundary condition, equation (9.79)(a), becomes

$$\frac{\rho_L\, g\, h^2}{6\, \mu_L} = \frac{\Delta P}{\mu_G\, L}\, \left(h^2 - D^2\right) + C_1\, (h - D) \tag{9.82}$$

The last integration constant, $C_1$ can be evaluated as

$$C_1 = \frac{\rho_L\, g\, h^2}{6\, \mu_L\, (h - D)} - \frac{\Delta P\, (h + D)}{\mu_G\, L} \tag{9.83}$$

With the integration constants evaluated, the gas velocity profile is

$$U_G = \frac{\Delta P}{\mu_G\, L}\, \left(x^2 - D^2\right) + \frac{\rho_L\, g\, h^2\, (x - D)}{6\, \mu_L\, (h - D)} - \frac{\Delta P\, (h + D)\, (x - D)}{\mu_G\, L} \tag{9.84}$$

The velocity in Equation (9.84) is equal to the velocity equation (9.64) when $(x = h)$. However, in that case, it is easy to show that the gas shear stress is not equal to the liquid shear stress at the interface (when the velocities are assumed to be the equal). The difference in shear stresses at the interface due to this assumption, of the equal velocities, cause this assumption to be not physical.

The second choice is to use the equal shear stresses at the interface, condition (9.79)(b). This condition requires that

$$\mu_G \frac{dU_G}{dx} = \mu_L \frac{dU_L}{dx} \tag{9.85}$$

The expressions for the derivatives are

$$\overbrace{\frac{2\,h\,\Delta P}{L} + \mu_G\,C_1}^{gas\ side} = \overbrace{\frac{2\,g\,h\,\rho_L}{3}}^{liquid\ side} \tag{9.86}$$

As result, the integration constant is

$$C_1 = \frac{2\,g\,h\,\rho_L}{3\,\mu_G} - \frac{2\,h\,\Delta P}{\mu_G\,L} \tag{9.87}$$

The gas velocity profile is then

$$U_G = \frac{\Delta P}{\mu_G\,L}\,\left(x^2 - D^2\right) + \left(\frac{2\,g\,h\,\rho_L}{3\,\mu_G} - \frac{2\,h\,\Delta P}{\mu_G\,L}\right)(x - D) \tag{9.88}$$

The gas velocity at the interface is then

$$U_G\big|_{@x=h} = \frac{\Delta P}{\mu_G\,L}\,\left(h^2 - D^2\right) + \left(\frac{2\,g\,h\,\rho_L}{3\,\mu_G} - \frac{2\,h\,\Delta P}{\mu_G\,L}\right)(h - D) \tag{9.89}$$

This gas interface velocity is different than the velocity of the liquid side. The velocity at interface can have a "slip" in very low density and for short distances. The shear stress at the interface must be equal, if no special effects occurs. Since there no possibility to have both the shear stress and velocity on both sides of the interface, different thing(s) must happen. It was assumed that the interface is straight but is impossible. Then if the interface becomes wavy, the two conditions can co–exist.

The wall shear stress is

$$\tau_G\big|_{@wall} = \mu_G\,\frac{dU_G}{dx}\bigg|_{x=D} = \mu_G\left(\frac{\Delta P\,2\,x}{\mu_G\,L} + \left(\frac{2\,g\,h\,\rho_L}{3\,\mu_G} - \frac{2\,h\,\Delta P}{\mu_G\,L}\right)\right)_{x=D} \tag{9.90}$$

or in a simplified form as

$$\tau_G\big|_{@wall} = \frac{2\,\Delta P\,(D-h)}{L} + \frac{2\,g\,h\,\rho_L}{3} \tag{9.91}$$

**The Required Pressure Difference**

The pressure difference to create the flooding (drying) has to take into account the fact that the surface is wavy. However, as first estimate the waviness of the surface can be neglected. The estimation of the pressure difference under the assumption of equal shear stress can be applied. In the same fashion the pressure difference under the assumption the equal velocity can be calculated. The actual pressure difference can be between these two assumptions but not must be between them. This model and its assumptions are too simplistic and the actual pressure difference is larger. However, this explanation is to show magnitudes and trends and hence it provided here.

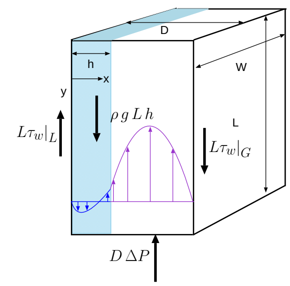


*Fig. -9.16. General forces diagram to calculated the in a two dimension geometry.*

To calculate the required pressure that cause the liquid to dry, the total balance is needed. The control volume include the gas and liquid volumes. Figure 9.16 describes the general forces that acts on the control volume. There are two forces that act against the gravity and two forces with the gravity. The gravity force on the gas can be neglected in most cases. The gravity force on the liquid is the liquid volume times the liquid volume as

$$F_{gL} = \rho\, g \overbrace{h\, L}^{Volme/w} \tag{9.92}$$

The total momentum balance is (see Figure 9.16)

$$F_{gL} + \overbrace{L}^{A/w} \tau_{w_G} = \overbrace{L}^{A/w} \tau_{w_L} + \overbrace{D\,\Delta P}^{force\ due\ to\ pressure} \tag{9.93}$$

Substituting the different terms into (9.93) result in

$$\rho\, g\, L\, h + L\left(\frac{2\,\Delta P\ (D-h)}{L} + \frac{2\, g\, h\, \rho_L}{3}\right) = L\,\frac{g\, h\, \rho_L}{3} + D\,\Delta P \tag{9.94}$$

Simplifying equation (9.94) results in

$$\frac{4\,\rho\, g\, L\, h}{3} = (2\,h - D)\,\Delta P \tag{9.95}$$

or

$$\Delta P = \frac{4\,\rho\, g\, L\, h}{3\,(2\,h - D)} \tag{9.96}$$

This analysis shows far more reaching conclusion that initial anticipation expected. The interface between the two liquid flowing together is wavy. Unless the derivations or assumptions are wrong, this analysis equation (9.96) indicates that when $D > 2\,h$ is a special case (extend open channel flow).

## 9.10 Multi–Phase Conclusion

For the first time multi–phase is included in a standard introductory textbook on fluid mechanics. There are several points that should be noticed in this chapter. There are many flow regimes in multi–phase flow that "regular" fluid cannot be used to solve it such as flooding. In that case, the appropriate model for the flow regime should be employed. The homogeneous models or combined models like Lockhart–Martinelli can be employed in some cases. In other case where more accurate measurement are needed a specific model is required. Perhaps as a side conclusion but important, the assumption of straight line is not appropriate when two liquid with different viscosity are flowing.

# APPENDIX A

# The Mathematics Backgrounds for Fluid Mechanics

In this appendix a review of selected topics in mathematics related to fluid mechanics is presented. These topics are present so that one with some minimal background could deal with the mathematics that encompass within basic fluid mechanics. Hence without additional reading, this book on fluid mechanics issues could be read by most readers. This appendix condenses material that spread in many various textbooks some of which are advance. Furthermore, some of the material appears in specialty books such as third order differential equations (and thus it is expected that the student is not familiar with this material.). There is very minimal original material which appears without proofs. The material is not presented in "educational" order but in importance order.

## A.1 Vectors

Vector is a quantity with direction as oppose to scalar. The length of the vector in Cartesian coordinates (the coordinates system is relevant) is

$$\|\boldsymbol{U}\| = \sqrt{{U_x}^2 + {U_y}^2 + {U_z}^2} \tag{A.1}$$

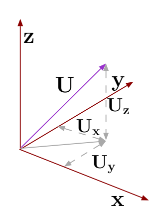


*Fig. -A.1. Vector in Cartesian coordinates system.*

Vector can be normalized and in Cartesian coordinates depicted in Figure A.1 where $U_x$ is the vector component in the $x$ direction, $U_y$ is the vector component in the $y$ direction, and $U_z$ is the vector component in the $z$ direction. Thus, the unit vector is

$$\widehat{\boldsymbol{U}} = \frac{\boldsymbol{U}}{\|\mathbf{U}\|} = \frac{U_x}{\|\boldsymbol{U}\|}\hat{\boldsymbol{i}} + \frac{U_y}{\|\boldsymbol{U}\|}\hat{\boldsymbol{j}} + \frac{U_z}{\|\boldsymbol{U}\|}\hat{\boldsymbol{k}} \tag{A.2}$$

and general orthogonal coordinates

$$\widehat{\boldsymbol{U}} = \frac{\boldsymbol{U}}{\|\mathbf{U}\|} = \frac{U_1}{\|\boldsymbol{U}\|}\mathbf{h_1} + \frac{U_2}{\|\boldsymbol{U}\|}\mathbf{h_2} + \frac{U_3}{\|\boldsymbol{U}\|}\mathbf{h_3} \tag{A.3}$$

Vectors have some what similar rules to scalars which will be discussed in the next section.

### A.1.1 Vector Algebra

Vectors obey several standard mathematical operations which are applicable to scalars. The following are vectors, $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ and for in this discussion $a$ and $b$ are scalars. Then the following can be said

1. $(\boldsymbol{U}+\boldsymbol{V})+\boldsymbol{W} = (\boldsymbol{U}+\boldsymbol{V}+\boldsymbol{W}) = \boldsymbol{U}+(\boldsymbol{V}+\boldsymbol{W})$
2. $\boldsymbol{U}+\boldsymbol{V} = \boldsymbol{V}+\boldsymbol{U}$
3. Zero vector is such that $\boldsymbol{U}+\mathbf{0} = \boldsymbol{U}$
4. Additive inverse $\boldsymbol{U}-\boldsymbol{U} = 0$
5. $a\,(\boldsymbol{U}+\boldsymbol{V}) = a\,\boldsymbol{U} + a\,\boldsymbol{V}$
6. $a\,(b\,\boldsymbol{U}) = a\,b\,\boldsymbol{U}$

The multiplications and the divisions have somewhat different meaning in a scalar operations. There are two kinds of multiplications for vectors. The first multiplication is the "dot" product which is defined by equation (A.4). The results of this multiplication is scalar but has no negative value as in regular scalar multiplication.

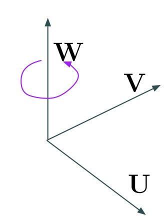


*Fig. -A.2. The right hand rule, multiplication of $\boldsymbol{U}\times\boldsymbol{V}$ results in $\boldsymbol{W}$.*

$$\boldsymbol{U}\cdot\boldsymbol{V} = \overbrace{|\boldsymbol{U}|\cdot|\boldsymbol{V}|}^{\text{regular scalar multiplication}}\ \cos\overbrace{(\angle(\boldsymbol{U},\boldsymbol{V}))}^{\text{angle between vectors}} \tag{A.4}$$

The second multiplication is the "cross" product which in vector as opposed to a scalar as in the "dot" product. The "cross" product is defined in an orthogonal coordinate ($\widehat{h_1}$, $\widehat{h_2}$, and $\widehat{h_3}$) as

$$\boldsymbol{U}\times\boldsymbol{V} = |\boldsymbol{U}|\cdot|\boldsymbol{V}|\sin\overbrace{(\angle(\boldsymbol{U},\boldsymbol{V}))}^{angle}\widehat{\boldsymbol{n}} \tag{A.5}$$

where $\theta$ is the angle between $\boldsymbol{U}$ and $\boldsymbol{V}$, and $\widehat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\boldsymbol{U}$ and $\boldsymbol{V}$ which obeys the right hand rule. The right hand rule is referred to the direction of resulting vector. Note that $\boldsymbol{U}$ and $\boldsymbol{V}$ are not necessarily orthogonal. Additionally note that order of multiplication is significant. This multiplication has a negative value which means that it is a change of the direction.

One of the consequence of this definitions in Cartesian coordinates is

$$\widehat{\boldsymbol{i}}^2 = \widehat{\boldsymbol{j}}^2 = \widehat{\boldsymbol{k}}^2 = 0 \tag{A.6}$$

In general for orthogonal coordinates this condition is written as

$$\widehat{\boldsymbol{h_1}} \times \widehat{\boldsymbol{h_1}} = \widehat{\boldsymbol{h_1}}^2 = \widehat{\boldsymbol{h_2}}^2 = \widehat{\boldsymbol{h_3}}^2 = 0 \tag{A.7}$$

where $\boldsymbol{h_i}$ is the unit vector in the orthogonal system.

In right hand orthogonal coordinate system

$$\begin{array}{ll}
\widehat{\boldsymbol{h_1}} \times \widehat{\boldsymbol{h_2}} = \widehat{\boldsymbol{h_3}} & \qquad \widehat{\boldsymbol{h_2}} \times \widehat{\boldsymbol{h_1}} = -\widehat{\boldsymbol{h_3}} \\
\widehat{\boldsymbol{h_2}} \times \widehat{\boldsymbol{h_3}} = \widehat{\boldsymbol{h_1}} & \qquad \widehat{\boldsymbol{h_3}} \times \widehat{\boldsymbol{h_2}} = -\widehat{\boldsymbol{h_1}} \\
\widehat{\boldsymbol{h_3}} \times \widehat{\boldsymbol{h_1}} = \widehat{\boldsymbol{h_2}} & \qquad \widehat{\boldsymbol{h_1}} \times \widehat{\boldsymbol{h_3}} = -\widehat{\boldsymbol{h_2}}
\end{array} \tag{A.8}$$

The "cross" product can be written as

$$\boldsymbol{U} \times \boldsymbol{V} = (U_2\, V_3 - U_3\, V_2)\,\widehat{\boldsymbol{h_1}} + (U_3\, V_1 - U_1\, V_3)\,\widehat{\boldsymbol{h_2}} + (U_1\, V_2 - U_2\, V_1)\,\widehat{\boldsymbol{h_3}} \tag{A.9}$$

Equation (A.9) in matrix form as

$$\boldsymbol{U} \times \boldsymbol{V} = \begin{pmatrix} \widehat{\boldsymbol{h_1}} & \widehat{\boldsymbol{h_2}} & \widehat{\boldsymbol{h_3}} \\ U_2 & U_2 & U_3 \\ V_2 & V_2 & V_3 \end{pmatrix} \tag{A.10}$$

The most complex of all these algebraic operations is the division. The multiplication in vector world have two definition one which results in a scalar and one which results in a vector. Multiplication combinations shows that there are at least four possibilities of combining the angle with scalar and vector. The reason that these current combinations, that is scalar associated with $\cos\theta$ vectors is associated with $\sin\theta$, is that these combinations have physical meaning. The previous experience is that help to define multiplication help to definition the division. The number of the possible combinations of the division is very large. For example, the result of the division can be a scalar combined or associated with the angle (with $\cos$ or $\sin$), or vector with the angle, etc. However, these above four combinations are not the only possibilities (not including the left hand system). It turn out that these combinations have very little[1] physical meaning. Additional possibility is that every combination of one vector element

[1]This author did find any physical meaning these combinations but there could be and those the word "little" is used.

is divided by the other vector element. Since every vector element has three possible elements the total combination is $9 = 3 \times 3$. There at least are two possibilities how to treat these elements. It turned out that combination of three vectors has a physical meaning. The three vectors have a need for additional notation such of vector of vector which is referred to as a tensor. The following combination is commonly suggested

$$\frac{\boldsymbol{U}}{\boldsymbol{V}} = \begin{pmatrix} \dfrac{U_1}{V_1} & \dfrac{U_2}{V_1} & \dfrac{U_3}{V_1} \\ \dfrac{U_1}{V_2} & \dfrac{U_2}{V_2} & \dfrac{U_3}{V_2} \\ \dfrac{U_1}{V_3} & \dfrac{U_2}{V_3} & \dfrac{U_3}{V_3} \end{pmatrix} \tag{A.11}$$

One such example of this division is the pressure which the explanation is commonality avoided or eliminated from the fluid mechanics books including the direct approach in this book.

This tenser or the matrix can undergo regular linear algebra operations such as finding the eigenvalue values and the eigen "vectors." Also note the multiplying matrices and inverse matrix are also available operation to these tensors.

### A.1.2 Differential Operators of Vectors

Differential operations can act on scalar functions as well on vector and vector functions. More differential operations can on scalar function can results in vector or vector function. In multivariate calculus, derivatives of different directions can represented as a vector or vector function. A compact presentation is a common way to handle the mathematics which simplify the calculations and explanations. One of these operations is nabla operator sometimes also called the "del operator." This operator is a differential vector. For example, in Cartesian coordinates the operation is

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \tag{A.12}$$

Where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are denoting unit vectors in the $x, y,$ and $z$ directions, respectively. Many of the operations of vector world, such as, the gradient, divergence, the curl, and the Laplacian are based or could be constructed from this single operator.

**Gradient**

This operation acts on a scalar function and results in a vector whose components are derivatives in the principle directions of a coordinate system. A scalar function is a function that provide a valued based on the coordinates (in Cartesian coordinates x,y,z). For example, the temperature of the domain might be expressed as a scalar field.

$$\nabla = \hat{\imath}\frac{\partial T}{\partial x} + \hat{\jmath}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z} \tag{A.13}$$

### Divergence

The same idea that was discussed in vector section there are two kinds of multiplication in the vector world and two will be for the differential operators. The divergence is the similar to "dot" product which results in scalar. A vector domain (function) assigns a vector to each point such as velocity for example, $N$, for Cartesian coordinates is

$$N(x,y,z) = N_x(x,y,z)\hat{\mathbf{i}} + N_y(x,y,z)\hat{\mathbf{j}} + N_z(x,y,z)\hat{\mathbf{k}} \tag{A.14}$$

The *dot* product of these two vectors, in Cartesian coordinate is results in

$$\mathrm{div}\ \mathbf{N} = \nabla \cdot \mathbf{N} = \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \frac{\partial N_z}{\partial z} \tag{A.15}$$

The divergence results in a scalar function which similar to the concept of the vectors multiplication of the vectors magnitude by the cosine of the angle between the vectors.

### Curl

Similar to the "cross product" a similar operation can be defined for the nabla (note the "right hand rule" notation) for Cartesian coordinate as

$$\begin{aligned}\mathrm{curl}\,\boldsymbol{N} = \nabla \times \boldsymbol{N} = \quad & \left(\frac{\partial N_z}{\partial y} - \frac{\partial N_y}{\partial z}\right)\hat{i} + \\ & \left(\frac{\partial N_x}{\partial z} - \frac{\partial N_z}{\partial x}\right)\hat{j} + \left(\frac{\partial N_y}{\partial x} - \frac{\partial N_x}{\partial y}\right)\hat{k}\end{aligned} \tag{A.16}$$

Note that the result is a vector.

### Laplacian

The new operation can be constructed from "dot" multiplication of the nabla. A gradient acting on a scalar field creates a vector field. Applying a divergence on the result creates a scalar field again. This combined operations is known as the "div grad" which is given in Cartesian coordinates by

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{A.17}$$

This combination is commonality denoted as $\nabla^2$. This operator also referred as the Laplacian operator, in honor of Pierre-Simon Laplace (23 March 1749 – 5 March 1827).

### d'Alembertian

As a super–set for four coordinates (very minimal used in fluid mechanics) and it reffed to as d'Alembertian or the wave operator, and it defined as

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t} \tag{A.18}$$

**Divergence Theorem**

Mathematicians call to or refer to a subset of The Reynolds Transport Theorem as the Divergence Theorem, or called it Gauss' Theorem (Carl Friedrich Gauss 30 April 1777 23 February 1855), In Gauss notation it is written as

$$\iiint_V (\nabla \cdot \boldsymbol{N})\, dV = \oiint_A \boldsymbol{N} \cdot \boldsymbol{n}\, dA \tag{A.19}$$

In Gauss-Ostrogradsky Theorem (Mikhail Vasilievich Ostrogradsky (September 24, 1801 – January 1, 1862). The notation is a bit different from Gauss and it is written in Ostrogradsky notation as

$$\int_V \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\, dy\, dz = \int \int_\Sigma (Pp + Qq + Rr)\, d\Sigma \tag{A.20}$$

Note the strange notation of "$\Sigma$" which refers to the area. This theorem is applicable for a fix control volume and the derivative can enters into the integral. Many engineering class present this theorem as a theorem on its merit without realizing that it is a subset of Reynolds Transport Theorem. This subset can further produces several interesting identities. If $\boldsymbol{N}$ is a gradient of a scalar field $\Pi(x, y, z)$ then it can insert into identity to produce

$$\iiint_V (\nabla \cdot (\nabla \boldsymbol{\Phi}))\, dV = \iiint_V \left(\nabla^2 \Phi\right) dV = \oiint_A \nabla \boldsymbol{\Phi} \cdot \boldsymbol{n}\, dA \tag{A.21}$$

Since the definition of $\nabla \boldsymbol{\Phi} = \boldsymbol{N}$.

Special case of equation (A.21) for harmonic function (solutions Laplace equation see[2] Harminic functions) then the left side vanishes which is useful identity for ideal flow analysis. This results reduces equation, normally for steady state, to a balance of the fluxes through the surface. Thus, the harmonic functions can be added or subtracted because inside the volume these functions contributions is eliminated throughout the volume.

### A.1.3 Differentiation of the Vector Operations

The vector operation sometime fell under (time or other) derivative. The basic of these relationships is explored. A vector is made of the several scalar functions such as

$$\vec{\boldsymbol{R}} = f_1(x_1, x_2, x_3, \cdots)\widehat{\boldsymbol{e}_1} + f_2(x_1, x_2, x_3, \cdots)\widehat{\boldsymbol{e}_2} + f_3(x_1, x_2, x_3, \cdots)\widehat{\boldsymbol{e}_3} + \cdots \tag{A.22}$$

where $\widehat{\boldsymbol{e}_i}$ is the unit vector in the $i$ direction. The cross and dot products when the come under differentiation can be look as scalar. For example, the dot product of operation

[2]for more information
http://math.fullerton.edu/mathews/c2003/HarmonicFunctionMod.html

$\boldsymbol{R}\cdot\boldsymbol{S} = (x\hat{i} + y^2\hat{j})\cdot(\sin x\hat{i} + exp(y)\hat{j})$ can be written as

$$\frac{d\,(\boldsymbol{R}\cdot\boldsymbol{S})}{dt} = \frac{d}{dt}\left(\left(x\hat{i} + y^2\hat{j}\right)\cdot\left(\sin x\hat{i} + exp(y)\hat{j}\right)\right)$$

It can be noticed that

$$\frac{d\,(\boldsymbol{R}\cdot\boldsymbol{S})}{dt} = \frac{d\left(x\,\sin x + y^2\,exp(y)\right)}{dt} = \frac{d\,x}{dt}\sin x + \frac{d\,\sin x}{dt} + \frac{d\,y^2}{dt}\exp(y) + \frac{d\,y^2}{dt}exp(y)$$

It can be noticed that the manipulation of the simple above example obeys the regular chain role. Similarly, it can done for the cross product. The results of operations of two vectors is similar to regular multiplication since the vectors operation obey "regular" addition and multiplication roles, the chain role is applicable. Hence the chain role apply for dot operation,

$$\frac{d}{dt}(\boldsymbol{R}\cdot\boldsymbol{S}) = \frac{d\boldsymbol{R}}{dt}\cdot\boldsymbol{S} + \frac{d\boldsymbol{S}}{dt}\cdot\boldsymbol{R} \tag{A.23}$$

And the the chain role for the cross operation is

$$\frac{d}{dt}(\boldsymbol{R}\times\boldsymbol{S}) = \frac{d\boldsymbol{R}}{dt}\times\boldsymbol{S} + \frac{d\boldsymbol{S}}{dt}\times\boldsymbol{R} \tag{A.24}$$

It follows that derivative (notice the similarity to scalar operations) of

$$\frac{d}{dt}(\boldsymbol{R}\cdot\boldsymbol{R}) = 2\,\boldsymbol{R}\frac{d\,\boldsymbol{R}}{at}$$

There are several identities that related to location, velocity, and acceleration. As in operation on scalar time derivative of dot or cross of constant velocity is zero. Yet, the most interesting is

$$\frac{d}{dt}(\boldsymbol{R}\times\boldsymbol{U}) = \boldsymbol{U}\times\boldsymbol{U} + \boldsymbol{R}\times\frac{d\boldsymbol{U}}{dt} \tag{A.25}$$

The first part is zero because the cross product with itself is zero. The second part is zero because Newton law (acceleration is along the path of R).

#### A.1.3.1 Orthogonal Coordinates

These vectors operations can appear in different orthogonal coordinates system. There are several orthogonal coordinates which appears in fluid mechanics operation which include this list: Cartesian coordinates, Cylindrical coordinates, Spherical coordinates, Parabolic coordinates, Parabolic cylindrical coordinates Paraboloidal coordinates, Oblate spheroidal coordinates, Prolate spheroidal coordinates, Ellipsoidal coordinates, Elliptic cylindrical coordinates, Toroidal coordinates, Bispherical coordinates, Bipolar cylindrical coordinates Conical coordinates, Flat-ring cyclide coordinates, Flat-disk cyclide coordinates, Bi-cyclide coordinates and Cap-cyclide coordinates. Because there are so many coordinates system is reasonable to develop these operations for any for any coordinates system. Three common systems typical to fluid mechanics will be presented and followed by a table and methods to present all the above equations.

### Cylindrical Coordinates

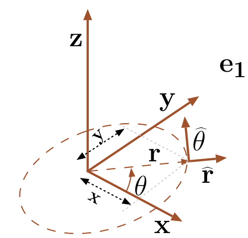


Fig. -A.3. Cylindrical Coordinate System.

The cylindrical coordinates are commonality used in situations where there is line of symmetry or kind of symmetry. This kind situations occur in pipe flow even if the pipe is not exactly symmetrical. These coordinates reduced the work, in most cases, because problem is reduced a two dimensions. Historically, these coordinate were introduced for geometrical problems about 2000 years ago[3]. The cylindrical coordinates are shown in Figure A.3. In the figure shows that the coordinates are $r$, $\theta$, and $z$. Note that unite coordinates are denoted as $\widehat{r}$, $\widehat{\theta}$, and $\widehat{z}$. The meaning of $\overrightarrow{r}$ and $\widehat{r}$ are different. The first one represents the vector that is the direction of $\widehat{r}$ while the second is the unit vector in the direction of the coordinate $r$. These three different $r$s are some what similar to any of the Cartesian coordinate. The second coordinate $\theta$ has unite coordinate $\widehat{\theta}$. The new concept here is the length factor. The coordinate $\theta$ is angle. In this book the dimensional chapter shows that in physics that derivatives have to have same units in order to compare them or use them. Conversation of the angel to units of length is done by length factor which is, in this case, $r$. The conversion between the Cartesian coordinate and the Cylindrical is

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan \frac{y}{x} \qquad z = z \tag{A.26}$$

The reverse transformation is

$$x = r\,\cos\theta \qquad y = r\,\sin\theta \qquad z = z \tag{A.27}$$

The line element and volume element are

$$ds = \sqrt{dr^2 + \left(r\,d\theta\right)^2 + dz^2} \qquad dr\,r\,d\theta\,dz \tag{A.28}$$

The gradient in cylindrical coordinates is given by

$$\nabla = \widehat{r}\frac{\partial}{\partial r} + \widehat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \widehat{z}\frac{\partial}{\partial z} \tag{A.29}$$

[3]Coolidge, Julian (1952). "The Origin of Polar Coordinates". American Mathematical Monthly 59: 7885. http://www-history.mcs.st-and.ac.uk/Extras/Coolidge_Polars.html. Note the advantage of cylindrical (polar) coordinates in description of geometry or location relative to a center point.

The curl is written

$$\nabla \times \boldsymbol{N} = \left( \frac{1}{r} \frac{\partial N_z}{\partial \theta} - \frac{\partial N_\theta}{\partial z} \right) \widehat{\boldsymbol{r}} + \left( \frac{\partial N_r}{\partial z} - \frac{\partial N_z}{\partial r} \right) \widehat{\boldsymbol{\theta}} + \tag{A.30}$$

$$\frac{1}{r} \left( \frac{\partial \left( r\, N_\theta \right)}{\partial r} - \frac{\partial N_\theta}{\partial \theta} \right) \widehat{\boldsymbol{z}} \tag{A.31}$$

The Laplacian is defined by

$$\nabla \cdot \nabla = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{A.32}$$

### Spherical Coordinates

The spherical coordinates system is a three-dimensional coordinates which is improvement or further modifications of the cylindrical coordinates. Spherical system used for cases where spherical symmetry exist. In fluid mechanics such situations exist in bubble dynamics, boom explosion, sound wave propagation etc. A location is represented by a radius and two angles. Note that the first angle (azimuth or longitude) $\theta$ range is between $0 < \theta < 2\pi$ while the second angle (colatitude) is only $0 < \phi < \pi$. The radius is the distance between the origin and the location. The first angle between projection on $x - y$ plane and the positive x–axis. The second angle is between the positive y–axis and the vector as shown in Figure A.4.

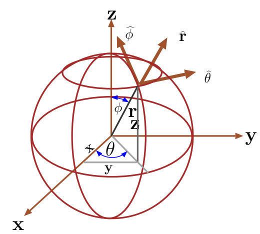


*Fig. -A.4. Spherical Coordinate System.*

The conversion between Cartesian coordinates to Spherical coordinates

$$x = r\,\sin\phi\,\cos\theta \qquad y = r\,\sin\phi\,\sin\theta \qquad z = r\,\cos\phi \tag{A.33}$$

The reversed transformation is

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \phi = \arccos\left(\frac{z}{r}\right) \tag{A.34}$$

Line element and element volume are

$$ds = \sqrt{dr^2 + \left(r\,\cos\theta\, d\theta\right)^2 + \left(r\,\sin\theta\, d\phi\right)^2} \qquad dV = r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{A.35}$$

The gradient is

$$\nabla = \widehat{\boldsymbol{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \widehat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \tag{A.36}$$

The divergence in spherical coordinate is

$$\nabla \cdot \boldsymbol{N} = \frac{1}{r^2} \frac{\partial \left( r^2 N_r \right)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial \left( N_\theta \sin\theta \right)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial N_\phi}{\partial \phi} \tag{A.37}$$

The curl in spherical coordinates is

$$\nabla \times \boldsymbol{N} = \frac{1}{r\sin\theta}\left(\frac{\partial\left(N_\phi \sin\theta\right)}{\partial\theta} - \frac{\partial N_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial N_r}{\partial\phi} - \frac{\partial\left(rN_\phi\right)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial\left(rN_\theta\right)}{\partial r} - \frac{\partial N_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}} \tag{A.38}$$

The Laplacian in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{A.39}$$

### General Orthogonal Coordinates

There are several orthogonal system and general form is needed. The notation for the presentation is required general notation of the units vectors is $\widehat{e_i}$ and coordinates distance coefficient is $h_i$ where $i$ is 1,2,3. The coordinates distance coefficient is the change the differential to the actual distance. For example in cylindrical coordinates, the unit vectors are: $\hat{r}$, $\hat{\theta}$, and $\hat{z}$. The units $\hat{r}$ and $\hat{z}$ are units with length. However, $\hat{\theta}$ is lengthens unit vector and the coordinate distance coefficient in this case is $r$. As in almost all cases, there is dispute what the proper notation for these coefficients. In mathematics it is denoted as $q$ while in engineering is denotes $h$. Since it is engineering book the $h$ is adapted. Also note that the derivative of the coordinate in the case of cylindrical coordinate is $\partial\theta$ and unit vector is $\hat{\theta}$. While the $\theta$ is the same the meaning is different and different notations need. The derivative quantity will be denoted by $q$ superscript.

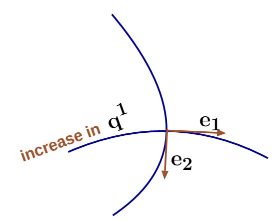


*Fig. -A.5. The general Orthogonal with unit vectors.*

The length of

$$d\ell^2 = \sum_{i=1}^{d}\left(h_k dq^k\right)^2 \tag{A.40}$$

The nabla operator in general orthogonal coordinates is

$$\nabla = \frac{\widehat{e_1}}{h_1}\frac{\partial}{\partial q^1} + \frac{\widehat{e_2}}{h_2}\frac{\partial}{\partial q^2} + \frac{\widehat{e_3}}{h_3}\frac{\partial}{\partial q^3} \tag{A.41}$$

### Gradient

The gradient in general coordinate for a scalar function $\boldsymbol{T}$ is the nabla operator in general orthogonal coordinates as

$$\nabla\boldsymbol{T} = \frac{\widehat{e_1}}{h_1}\frac{\partial\boldsymbol{T}}{\partial q^1} + \frac{\widehat{e_2}}{h_2}\frac{\partial\boldsymbol{T}}{\partial q^2} + \frac{\widehat{e_3}}{h_3}\frac{\partial\boldsymbol{T}}{\partial q^3} \tag{A.42}$$

The divergence of a vector equals

$$\nabla \cdot \boldsymbol{N} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q^1}\left(N_1 h_2 h_3\right) + \frac{\partial}{\partial q^2}\left(N_2 h_3 h_1\right) + \frac{\partial}{\partial q^3}\left(N_3 h_1 h_2\right)\right]. \tag{A.43}$$

For general orthogonal coordinate system the curl is

$$\begin{gathered}\nabla \times \boldsymbol{N} = \frac{\widehat{e_1}}{h_2\, h_3}\left[\frac{\partial}{\partial q^2}\left(h_3\, N_3\right) - \frac{\partial}{\partial q^3}\left(h_2\, N_2\right)\right] + \\ \frac{\widehat{e_2}}{h_3\, h_1}\left[\frac{\partial}{\partial q^3}\left(h_1\, N_1\right) - \frac{\partial}{\partial q^1}\left(h_3\, N_3\right)\right] + \frac{\widehat{e_3}}{h_1\, h_2}\left[\frac{\partial}{\partial q^1}\left(h_2 N_2\right) - \frac{\partial}{\partial q^2}\left(h_1 N_1\right)\right]\end{gathered} \tag{A.44}$$

The Laplacian of a scalar equals

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q^1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial q^1}\right) + \frac{\partial}{\partial q^2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial q^2}\right) + \frac{\partial}{\partial q^3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial q^3}\right)\right] \tag{A.45}$$

The following table showing the different values for selected orthogonal system.

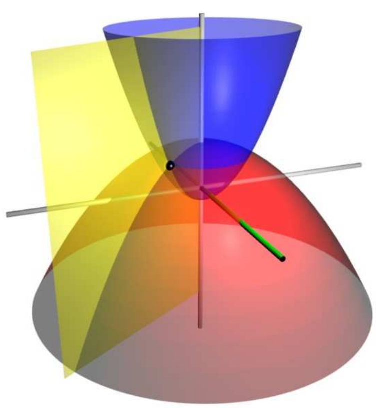

*Fig. -A.6. Parabolic coordinates by user WillowW using Blender.*

*Table -A.1. Orthogonal coordinates systems (under construction please ignore)*

| **Orthogonal coordinates systems** | Remarks | $h$ | | | $q$ | | |
|---|---|---|---|---|---|---|---|
| **name** | | 1 | 2 | 3 | 1 | 2 | 3 |
| **Cartesian** | standard | $1$ | $1$ | $1$ | $x$ | $y$ | $z$ |
| **Cylindrical** | common | $1$ | $r$ | $1$ | $r$ | $\theta$ | $z$ |
| **Spherical** | common | $1$ | $r$ | $r\cos\theta$ | $r$ | $\theta$ | $\varphi$ |
| **Paraboloidal** | ? | $\sqrt{u^2+v^2}$ | $\sqrt{u^2+v^2}$ | $u\,v$ | $u$ | $v$ | $\theta$ |
| **Ellipsoidal** | ? | | | | $\lambda$ | $\mu$ | $\nu$ |

## A.2 Ordinary Differential Equations (ODE)

In this section a brief summary of ODE is presented. It is not intent to be a replacement to a standard textbook but as a quick reference. It is suggested that the reader interested in depth information should read "Differential Equations and Boundary Value Problems" by Boyce de–Prima or any other book in this area. Ordinary differential equations are defined by the order of the highest derivative. If the highest derivative is first order the equation is referred as first order differential equation etc. Note that the derivatives are integers e.g. first derivative, second derivative etc[4]. ODE are categorized into linear and non-linear equations. The meaning of linear equation is that the operation is such that

$$a\,L\,(u_1) + b\,L\,(u_2) = L\,(a\,u_1 + b\,u_2) \tag{A.46}$$

An example of such linear operation $L = \frac{d}{dt} + 1$ acting on $y$ is $\frac{dy_1}{dt} + y_1$. Or this operation on $y_2$ is $\frac{dy_2}{dt} + y_2$ and the summation of operation the sum operation of $L(y_1 + y_2) = \frac{y_1+y_2}{dt} + y_1 + y_2$.

### A.2.1 First Order Differential Equations

As expect, the first ODEs are easier to solve and they are the base for equations of higher order equation. The first order equations have several forms and there is no one solution fit all but families of solutions. The most general form is

$$f\left(u, \frac{du}{dt}, t\right) = 0 \tag{A.47}$$

[4]Note that mathematically, it is possible to define fraction of derivative. However, there is no physical meaning to such a product according to this author believe.

Sometimes equation (A.47) can be simplified to the first form as

$$\frac{du}{dt} = F(t,u) \tag{A.48}$$

### A.2.2 Variables Separation or Segregation

In some cases equation (A.48) can be written as $F(t,u) = X(t)\,U(u)$. In that case it is said that $F$ is spreadable and then equation (A.48) can be written as

$$\frac{du}{U(u)} = X(t)dt \tag{A.49}$$

Equation can be integrated either analytically or numerically and the solution is

$$\boxed{\int \frac{du}{U(u)} = \int X(t)dt} \tag{A.50}$$

The limits of the integral is (are) the initial condition(s). The initial condition is the value the function has at some points. The name initial condition is used because the values are given commonly at initial time.

Example A.1:
*Solve the following equation*

$$\frac{du}{dt} = u\,t \tag{1.I.a}$$

*with the initial condition* $u(t=0) = u_0$.

SOLUTION

The solution can be obtained by the variable separation method. The separation yields

$$\frac{du}{u} = t\,dt \tag{1.I.b}$$

The integration of equation (1.I.b) becomes

$$\int \frac{du}{u} = \int t\,dt \Longrightarrow \ln(u) + \ln(c) = \frac{t^2}{2} \tag{1.I.c}$$

Equation (1.I.c) can be transferred to

$$u = c\,e^{t^2} \tag{1.I.d}$$

For the initial condition of $u(0) = u_0$ then

$$u = u_0\,e^{t^2} \tag{1.I.e}$$

End Solution

#### A.2.2.1 The Integral Factor Equations

Another method is referred to as integration factor which deals with a limited but very important class of equations. This family is part of a linear equations. The general form of the equation is

$$\frac{dy}{dx} + g(x)\, y = m(x) \tag{A.51}$$

Multiplying equation (A.51) by unknown function $N(x)$ transformed it to

$$N(x)\,\frac{dy}{dx} + N(x)\, g(x)\, y = N(x)m(x) \tag{A.52}$$

What is needed from $N(x)$ is to provide a full differential such as

$$N(x)\,\frac{dy}{dx} + N(x)\, g(x)\, y = \frac{d\,[N(x)\, g(x)\, y]}{dx} \tag{A.53}$$

This condition (note that the previous methods is employed here) requires that

$$\frac{d\,N(x)}{dx} = N(x)\, g(x) \Longrightarrow \frac{d\,N(x)}{N(x)} = g(x)\, dx \tag{A.54}$$

Equation (A.54) is integrated to be

$$\ln\left(N(x)\right) = \int g(x) dx \Longrightarrow N(x) = \mathsf{e}^{\displaystyle\int g(x) dx} \tag{A.55}$$

Using the differentiation chain rule provides

$$\frac{d\,N(x)}{dx} = \overbrace{\mathsf{e}^{\displaystyle\int g(x) dx}}^{\frac{dv}{du}}\overbrace{g(x)}^{\frac{du}{dx}} \tag{A.56}$$

which indeed satisfy equation (A.53). Thus equation (A.52) becomes

$$\frac{d\,[N(x)\, g(x)\, y]}{dx} = N(x)\, m(x) \tag{A.57}$$

Multiplying equation (A.57) by $dx$ and integrating results in

$$N(x)\, g(x)\, y = \int N(x)\, m(x)\, dx \tag{A.58}$$

The solution is then

$$\boxed{y = \frac{\displaystyle\int N(x)\, m(x)\, dx}{g(x)\, \underbrace{\mathsf{e}^{\int g(x) dx}}_{N(x)}}} \tag{A.59}$$

A special case of $g(t) = constant$ is shown next.

Example A.2:
*Find the solution for a typical problem in fluid mechanics (the problem of Stoke flow or the parachute problem) of*

$$\frac{dy}{dx} + y = 1$$

SOLUTION

Substituting $m(x) = 1$ and $g(x) = 1$ into equation (A.59) provides

$$y = \mathsf{e}^{-x}\ (\mathsf{e}^{x} + c) = 1 + c\,\mathsf{e}^{-x}$$

End Solution

### A.2.3 Non–Linear Equations

Non-Linear equations are equations that the power of the function or the function derivative is not equal to one or their combination. Many non linear equations can be transformed into linear equations and then solved with the linear equation techniques. One such equation family is referred in the literature as the Bernoulli Equations[5]. This equation is

$$\frac{du}{dt} + m(t)u = n(t) \overbrace{u^p}^{\text{non–linear part}} \tag{A.60}$$

The transformation $v = u^{1-p}$ turns equation (A.60) into a linear equation which is

$$\frac{dv}{dt} + (1-p)\,m(t)\,v = (1-p)\,n(t) \tag{A.61}$$

The linearized equation can be solved using the linear methods. The actual solution is obtained by reversed equation which transferred solution to

$$u = v^{(p-1)} \tag{A.62}$$

Example A.3:
*Solve the following Bernoulli equation*

$$\frac{du}{dt} + t^2\,u = \sin(t)\,u^3 \tag{1.III.a}$$

[5]Not to be confused with the Bernoulli equation without the $s$ that referred to the energy equation.

Solution

The transformation is

$$v = u^2 \tag{1.III.b}$$

Using the definition (1.III.b) equation (1.III.a) becomes

$$\frac{dv}{dt} \overbrace{-2}^{1-p} t^2\, v = \overbrace{-2}^{1-p} sin(t) \tag{1.III.c}$$

The homogeneous solution of equation (1.III.c) is

$$u(t) = c e^{\frac{-t^3}{3}} \tag{1.III.d}$$

And the general solution is

$$u = e^{-\frac{t^3}{3}} \left( \overbrace{\int e^{\frac{t^3}{3}} \sin(t)\, dt}^{\text{private solution}} + c \right) \tag{1.III.e}$$

End Solution

#### A.2.3.1 Homogeneous Equations

Homogeneous function is given as

$$\frac{du}{dt} = f(u,t) = f(a\,u, a\,t) \tag{A.63}$$

for any real positive $a$. For this case, the transformation of $u = v\,t$ transforms equation (A.63) into

$$t\,\frac{dv}{dt} + v = f(1,v) \tag{A.64}$$

In another words if the substitution $u = v\,t$ is inserted the function $f$ become a function of only $v$ it is homogeneous function. Example of such case $u\prime = \left(u^3 - t^3\right)/t^3$ becomes $u\prime = \left(v^3 + 1\right)$. The solution is then

$$\ln|t| = \int \frac{dv}{f(1,v) - v} + c \tag{A.65}$$

Example A.4:

*Solve the equation*

$$\frac{du}{dt} = \sin\left(\frac{u}{t}\right) + \left(\frac{u^4 - t^4}{t^4}\right) \tag{1.IV.a}$$

SOLUTION

Substituting $u = v\,T$ yields

$$\frac{du}{dt} = \sin(v) + v^4 - 1 \tag{1.IV.b}$$

or

$$t\,\frac{dv}{dt} + v = \sin(v) + v^4 - 1 \Longrightarrow t\,\frac{dv}{dt} = \sin(v) + v^4 - 1 - v \tag{1.IV.c}$$

Now equation (1.IV.c) can be solved by variable separation as

$$\frac{dv}{\sin(v) + v^4 - 1 - v} = t\,dt \tag{1.IV.d}$$

Integrating equation (1.IV.d) results in

$$\int \frac{dv}{\sin(v) + v^4 - 1 - v} = \frac{t^2}{2} + c \tag{1.IV.e}$$

The initial condition can be inserted via the boundary of the integral.

End Solution

#### A.2.3.2 Variables Separable Equations

In fluid mechanics and many other fields there are differential equations that referred to variables separable equations. In fact, this kind of class of equations appears all over this book. For this sort equations, it can be written that

$$\frac{du}{dt} = f(t)g(u) \tag{A.66}$$

The main point is that $f(t)$ and be segregated from $g(u)$. The solution of this kind of equation is

$$\boxed{\int \frac{du}{g(u)} = \int f(t)\,dt} \tag{A.67}$$

Example A.5:

*Solve the following ODE*

$$\frac{du}{dt} = -u^2\,t^2 \tag{1.V.a}$$

SOLUTION

Segregating the variables to be

$$\int \frac{du}{u^2} = \int t^2\,dt \tag{1.V.b}$$

Integrating equation (1.V.b) transformed into

$$-\frac{1}{u} = \frac{t^3}{3} + c_1 \tag{1.V.c}$$

Rearranging equation (1.V.c) becomes

$$u = \frac{-3}{t^3 + c} \tag{1.V.d}$$

End Solution

#### A.2.3.3 Other Equations

There are equations or methods that were not covered by the above methods. There are additional methods such numerical analysis, transformation (like Laplace transform), variable substitutions, and perturbation methods. Many of these methods will be eventually covered by this appendix.

### A.2.4 Second Order Differential Equations

The general idea of solving second order ODE is by converting them into first order ODE. One such case is the second order ODE with constant coefficients.

The simplest equations are with constant coefficients such as

$$a\,\frac{d^2u}{dt^2} + b\,\frac{du}{dt} + c\,u = 0 \tag{A.68}$$

In a way, the second order ODE is transferred to first order by substituting the one linear operator to two first linear operators. Practically, it is done by substituting $e^{st}$ where $s$ is characteristic constant and results in the quadratic equation

$$a\,s^2 + b\,s + s = 0 \tag{A.69}$$

If $b^2 > 4\,a\,c$ then there are two unique solutions for the quadratic equation and the general solution form is

$$u = c_1\,e^{s_1\,t} + c_2\,e^{s_2\,t} \tag{A.70}$$

For the case of $b^2 = 4\,a\,c$ the general solution is

$$u = c_1\,e^{s_1\,t} + c_2\,t\,e^{s_1\,t} \tag{A.71}$$

In the case of $b^2 > 4\,a\,c$, the solution of the quadratic equation is a complex number which means that the solution has exponential and trigonometric functions as

$$u = c_1 e^{\alpha\,t}\cos(\beta t) + c_2 e^{\alpha\,t}\sin(\beta t) \tag{A.72}$$

Where the real part is

$$\alpha = \frac{-b}{2\,a} \tag{A.73}$$

and the imaginary number is

$$\beta = \frac{\sqrt{4\,a\,c - b^2}}{2\,a} \tag{A.74}$$

Example A.6:
*Solve the following ODE*

$$\frac{d^2u}{dt^2} + 7\,\frac{du}{dt} + 10\,u = 0 \tag{1.VI.a}$$

Solution

The characteristic equation is

$$s^2 + 7\,s + 10 = 0 \tag{1.VI.b}$$

The solution of equation (1.VI.b) are $-2$, and $-5$. Thus, the solution is

$$u = k_1\,e^{-2\,t} + k_2\,e^{-5\,t} \tag{1.VI.c}$$

End Solution

#### A.2.4.1 Non–Homogeneous Second ODE

Homogeneous equation are equations that equal to zero. This fact can be used to solve non-homogeneous equation. Equations that not equal to zero in this form

$$a\,\frac{d^2u}{dt^2} + b\,\frac{du}{dt} + c\,u = l(x) \tag{A.75}$$

The solution of the homogeneous equation is zero that is the operation $L(u_h) = 0$, where $L$ is Linear operator. The additional solution of $L(u_p)$ is the total solution as

$$L\left(u_{total}\right) = \overbrace{L\left(u_h\right)}^{=0} + L\left(u_p\right) \Longrightarrow u_{total} = u_h + u_p \tag{A.76}$$

Where the solution $u_h$ is the solution of the homogeneous solution and $u_p$ is the solution of the particular function $l(x)$. If the function on the right hand side is polynomial than the solution is will

$$u_{total} = u_h + \sum_{i=1}^{n} u_{p_i} \tag{A.77}$$

The linearity of the operation creates the possibility of adding the solutions.

Example A.7:
*Solve the non-homogeneous equation*

$$\frac{d^2u}{dt^2} - 5\,\frac{du}{dt} + 6\,u = t + t^2$$

SOLUTION

The homogeneous solution is

$$u(t) = c_1\,e^{2\,t} + c_1\,e^{3\,t} \tag{1.VII.a}$$

the particular solution for $t$ is

$$u(t) = \frac{6\,t + 5}{36} \tag{1.VII.b}$$

and the particular solution of the $t^2$ is

$$u(t) = \frac{18\,t^2 + 30\,t + 19}{108} \tag{1.VII.c}$$

The total solution is

$$u(t) = c_1\,e^{2\,t} + c_1\,e^{3\,t} + \frac{9\,t^2 + 24\,t + 17}{54} \tag{1.VII.d}$$

End Solution

### A.2.5 Non–Linear Second Order Equations

Some of the techniques that were discussed in the previous section (first order ODE) can be used for the second order ODE such as the variable separation.

#### A.2.5.1 Segregation of Derivatives

If the second order equation

$$f(u, \dot{u}, \ddot{u}) = 0$$

can be written or presented in the form

$$f_1(u)\dot{u} = f_2\left(\dot{u}\right)\ddot{u} \tag{A.78}$$

then the equation (A.78) is referred to as a separable equation (some called it segregated equations). The derivative of $\dot{u}$ can be treated as a new function $v$ and $\dot{v} = \ddot{u}$. Hence, equation (A.78) can be integrated

$$\int_{u_0}^{u} f_1(u)\dot{u} = \int_{\dot{u}_0}^{\dot{u}} f_2\left(\dot{u}\right)\ddot{u} = \int_{v_0}^{v} f_2\left(u\right)\dot{v} \tag{A.79}$$

The integration results in a first order differential equation which should be dealt with the previous methods. It can be noticed that the function initial condition is used twice; first with initial integration and second with the second integration. Note that the derivative initial condition is used once. The physical reason is that the equation represents a strong effect of the function at a certain point such surface tension problems. This equation family is not well discussed in mathematical textbooks[6].

Example A.8:

*Solve the equation*

$$\sqrt{u}\,\frac{du}{dt} - \sin\left(\frac{du}{dt}\right)\frac{d^2u}{dt^2} = 0$$

*With the initial condition of* $u(0) = 0$ *and* $\frac{du}{dt}(t=0) = 0$ *What happen to the extra "dt"?*

<u>Solution</u>

Rearranging the ODE to be

$$\sqrt{u}\frac{du}{\cancel{dt}} = \sin\left(\frac{du}{dt}\right)\frac{d}{\cancel{dt}}\left(\frac{du}{dt}\right) \tag{1.VIII.a}$$

Thus the extra $dt$ is disappeared and equation (1.VIII.a) becomes

$$\int\sqrt{u}\,du = \int\sin\left(\frac{du}{dt}\right)d\left(\frac{du}{dt}\right) \tag{1.VIII.b}$$

and transformation to $v$ is

$$\int\sqrt{u}\,du = \int\sin(v)\,dv \tag{1.VIII.c}$$

After the integration equation (1.VIII.c) becomes

$$\frac{2}{3}\left(u^{\frac{3}{2}} - {u_0}^{\frac{3}{2}}\right) = \cos(v_0) - \cos(v) = \cos\left(\frac{du_0}{dt}\right) - \cos\left(\frac{du}{dt}\right) \tag{1.VIII.d}$$

Equation (1.VIII.d) can be rearranged as

$$\frac{du}{dt} = \arcsin\left(\frac{2}{3}\left({u_0}^{\frac{3}{2}} - u^{\frac{3}{2}}\right) + \cos(v_0)\right) \tag{A.80}$$

Using the first order separation method yields

$$\int_0^t dt = \int_{u_0}^{u}\frac{du}{\arcsin\left(\frac{2}{3}\left(\underbrace{{u_0}^{\frac{3}{2}}}_{=0} - u^{\frac{3}{2}}\right) + \underbrace{\cos(v_0)}_{=1}\right)} \tag{A.81}$$

[6]This author worked (better word toyed) in (with) this area during his master but to his shame he did not produce any papers on this issue. The papers are still his drawer and waiting to a spare time.

The solution (A.81) shows that initial condition of the function is used twice while the initial of the derivative is used only once.

End Solution

#### A.2.5.2 Full Derivative Case Equations

Another example of special case or families of second order differential equations which is results of the energy integral equation derivations as

$$u - a\,u \left(\frac{du}{d\,t}\right)\left(\frac{d^{2u}}{d\,t^2}\right) = 0 \tag{A.82}$$

where $a$ is constant. One solution is $u = k_1$ and the second solution is obtained by solving

$$\frac{1}{a} = \left(\frac{du}{d\,t}\right)\left(\frac{d^{2u}}{d\,t^2}\right) \tag{A.83}$$

The transform of $v = \frac{du}{dt}$ results in

$$\frac{1}{a} = v\frac{dv}{dt} \Longrightarrow \frac{dt}{a} = v\,dv \tag{A.84}$$

which can be solved with the previous methods.

Bifurcation to two solutions leads

$$\frac{t}{a} + c = \frac{1}{2}\,v^2 \Longrightarrow \frac{du}{dt} = \pm\sqrt{\frac{2\,t}{a} + c_1} \tag{A.85}$$

which can be integrated as

$$u = \int \pm\sqrt{\frac{2\,t}{a} + c_1}\;dt = \pm\frac{a}{3}\left(\frac{2\,t}{a} + c_1\right)^{\frac{3}{2}} + c_2 \tag{A.86}$$

#### A.2.5.3 Energy Equation ODE

It is non–linear because the second derivative is square and the function multiply the second derivative.

$$u\left(\frac{d^2u}{d\,t^2}\right) + \left(\frac{du}{d\,t}\right)^2 = 0 \tag{A.87}$$

It can be noticed that that $c_2$ is actually two different constants because the plus minus signs.

$$\frac{d}{dt}\left(u\,\frac{du}{d\,t}\right) = 0 \tag{A.88}$$

after integration

$$u \frac{du}{d\,t} = k_1 \tag{A.89}$$

Further rearrangement and integration leads to the solution which is

$$\frac{u^2}{2\,k_1} = t + k_2 \tag{A.90}$$

For non–homogeneous equation they can be integrated as well.

Example A.9:
*Show that the solution of*

$$u \left(\frac{d^2u}{d\,t^2}\right) + \left(\frac{du}{d\,t}\right)^2 + u = 0 \tag{1.IX.a}$$

*is*

$$-\frac{\sqrt{3}\displaystyle\int \frac{u}{\sqrt{3\,k_1 - u^3}} du}{\sqrt{2}} = t + k_2 \tag{1.IX.b}$$

$$\frac{\sqrt{3}\displaystyle\int \frac{u}{\sqrt{3\,k_1 - u^3}} du}{\sqrt{2}} = t + k_2 \tag{1.IX.c}$$

### A.2.6 Third Order Differential Equation

There are situations where fluid mechanics[7] leads to third order differential equation. This kind of differential equation has been studied in the last 30 years to some degree. The solution to constant coefficients is relatively simple and will be presented here. Solution to more complicate linear equations with non constant coefficient (function of $t$) can be solved sometimes by Laplace transform or reduction of the equation to second order Olivier Vallee[8].

The general form for constant coefficient is

$$\frac{d^3u}{dt^3} + a\,\frac{d^2u}{dt^2} + b\,\frac{du}{dt} + c\,u = 0 \tag{A.91}$$

The solution is assumed to be of the form of $e^{st}$ which general third order polonium. Thus, the general solution is depend on the solution of third order polonium. Third

[7]The unsteady energy equation in accelerated coordinate leads to a third order differential equation.

[8]"On the linear third-order differential equation" Springer Berlin Heidelberg, 1999. Solving Third Order Linear Differential Equations in Terms of Second Order Equations Mark van Hoeij

order polonium has always one real solution. Thus, derivation of the leading equation (results of the ode) is reduced into quadratic equation and thus the same situation exist.

$$s^3 + a_1\, s^2 + a_2\, s + a_3 = 0 \tag{A.92}$$

The solution is

$$s_1 = -\frac{1}{3}a_1 + (S+T) \tag{A.93}$$

$$s_2 = -\frac{1}{3}a_1 - \frac{1}{2}(S+T) + \frac{1}{2}i\sqrt{3}(S-T) \tag{A.94}$$

and

$$s_3 = -\frac{1}{3}a_1 - \frac{1}{2}(S+T) - \frac{1}{2}i\sqrt{3}(S-T) \tag{A.95}$$

Where

$$S = \sqrt[3]{R + \sqrt{D}}, \tag{A.96}$$

$$T = \sqrt[3]{R - \sqrt{D}} \tag{A.97}$$

and where the $D$ is defined as

$$D = Q^3 + R^2 \tag{A.98}$$

and where the definitions of $Q$ and $R$ are

$$Q = \frac{3a_2 - {a_1}^2}{9} \tag{A.99}$$

and

$$R = \frac{9a_1a_2 - 27a_3 - 2{a_1}^3}{54} \tag{A.100}$$

Only three roots can exist for the Mach angle, $\theta$. From a mathematical point of view, if $D > 0$, one root is real and two roots are complex. For the case $D = 0$, all the roots are real and at least two are identical. In the last case where $D < 0$, all the roots are real and unequal.

When the characteristic equation solution has three different real roots the solution of the differential equation is

$$u = c_1\, e^{s_1\, t} + c_2\, e^{s_2\, t} + c_3\, e^{s_3\, t} \tag{A.101}$$

In the case the solution to the characteristic has two identical real roots

$$u = (c_1 + c_2\, t)\, e^{s_1\, t} + c_3\, e^{s_2\, t} \tag{A.102}$$

Similarly derivations for the case of three identical real roots. For the case of only one real root, the solution is

$$u = (c_1 \sin b_1 + c_2 \cos b_1)\, e^{a_1 t} + c_3 e^{s_3 t} \tag{A.103}$$

Where $a_1$ is the real part of the complex root and $b_1$ imaginary part of the root.

### A.2.7 Forth and Higher Order ODE

The ODE and partial differential equations (PDE) can be of any integer order. Sometimes the ODE is fourth order or higher the general solution is based in idea that equation is reduced into a lower order. Generally, for constant coefficients ODE can be transformed into multiplication of smaller order linear operations. For example, the equation

$$\frac{d^4 u}{dt^4} - u = 0 \Longrightarrow \left(\frac{d^4}{dt^4} - 1\right) u = 0 \tag{A.104}$$

can be written as combination of

$$\left(\frac{d^2}{dt^2} - 1\right)\left(\frac{d^2}{dt^2} + 1\right) u = 0 \quad \text{or} \quad \left(\frac{d^2}{dt^2} + 1\right)\left(\frac{d^2}{dt^2} - 1\right) u = 0 \tag{A.105}$$

The order of operation is irrelevant as shown in equation (A.105). Thus the solution of

$$\left(\frac{d^2}{dt^2} + 1\right) u = 0 \tag{A.106}$$

with the solution of

$$\left(\frac{d^2}{dt^2} - 1\right) u = 0 \tag{A.107}$$

are the solutions of (A.104). The solution of equation (A.106) and equation (A.107) was discussed earlier.

The general procedure is based on the above concept but is some what simpler. Inserting $e^{s\,t}$ into the ODE

$$a_n\, u^{(n)} + a_{n-1}\, u^{(n-1)} + a_{n-2}\, u^{(n-2)} + \cdots + a_1\, u' + a_0\, u = 0 \tag{A.108}$$

yields characteristic equation

$$a_n\, s^n + a_{n-1}\, s^{n-1} + a_{n-2}\, s^{n-2} + \cdots + a_1\, s + a_0 = 0 \tag{A.109}$$

| **If The Solution of Characteristic Equation** | **The Solution of Differential Equation Is** |
|---|---|
| all roots are real and different e.g. $s_1 \neq s_2 \neq s_3 \neq s_4 \cdots \neq s_n$ | $u = c_1\, e^{s_1 t} + c_2\, e^{s_2 t} + \cdots + c_n\, e^{s_n t}$ |
| all roots are real but some are identical e.g. $s_1 = s_2 = \cdots = s_k$ and some different e.g. $s_{k+1} \neq s_{k+2} \neq s_{k+3} \cdots \neq s_n$ | $u = \left(c_1 + c_2\, t + \cdots + c_k\, t^{k-1}\right)\, e^{s_1 t} +$ $c_{k+1}\, e^{s_{k+1} t} + c_{k+2}\, e^{s_{k+2} t} + \cdots + c_n\, e^{s_n t}$ |
| $k/2$ roots, are pairs of conjugate complex numbers of $s_i = a_i \pm b_i$ and some real and different e.g. $s_{k+1} \neq s_{k+2} \neq s_{k+3} \cdots \neq s_n$ | $u = (\cos(b_1\, t) + \sin(b_1\, t))\, e^{a_1 t} +$ $\cdots + (\cos(b_i\, t) + \sin(b_i\, t))\, e^{a_i t} +$ $\cdots + (\cos(b_k\, t) + \sin(b_k\, t))\, e^{a_k t} +$ $c_{k+1}\, e^{s_{k+1} t} + c_{k+2}\, e^{s_{k+2} t} + \cdots + c_n\, e^{s_n t}$ |
| $k/2$ roots, are pairs of conjugate complex numbers of $s_i = a_i \pm b_i$, $\ell$ roots are similar and some real and different e.g. $s_{k+1} \neq s_{k+2} \neq s_{k+3} \cdots \neq s_n$ | $u = (\cos(b_1\, t) + \sin(b_1\, t))\, e^{a_1 t} +$ $\cdots + (\cos(b_i\, t) + \sin(b_i\, t))\, e^{a_i t} +$ $\cdots + (\cos(b_k\, t) + \sin(b_k\, t))\, e^{a_k t} +$ $\left(c_{k+1} + c_{k+2}\, t + \cdots + c_{k+\ell}\, t^{\ell-1}\right)\, e^{s_{k+1} t} +$ $c_{k+2}\, e^{s_{k+2} t} + c_{k+3}\, e^{s_{k+3} t} + \cdots + c_n\, e^{s_n t}$ |

Example A.10:

*Solve the fifth order ODE*

$$\frac{d^5 u}{dt^5} - 11\, \frac{d^4 u}{dt^4} + 57\, \frac{d^3 u}{dt^3} - 149\, \frac{d^2 u}{dt^2} + 192\, \frac{du}{dt} - 90\, u = 0 \tag{1.X.a}$$

<u>Solution</u>

The characteristic equation is

$$s^5 - 11\, s^4 + 57\, s^3 - 149\, s^2 + 192\, s - 90 = 0 \tag{1.X.b}$$

With the roots of the equation (1.X.b) (these roots can be found using numerical methods or Descartes' Rule) are

$$\begin{aligned} s_{1,2} &= 3 \pm 3\, i \\ s_{3,4} &= 2 \pm i \\ s_5 &= 1 \end{aligned} \tag{1.X.c}$$

The roots are two pairs of complex numbers and one real number. Thus the solution is

$$u = c_1\, e^t + e^{2\,t}\, \left(c_2\, \sin(t) + c_3\, \cos(t)\right) + e^{3\,t}\, \left(c_4\, \sin(3\,t) + c_5\, \cos(3\,t)\right) \tag{1.X.d}$$

End Solution

### A.2.8 A general Form of the Homogeneous Equation

The homogeneous equation can be generalized to

$$k_0\, t^n \frac{d^n u}{dt^n} + k_1\, t^{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \cdots + k_{n-1}\, t \frac{du}{dt} + k_n\, u = a\, x \tag{A.110}$$

To be continue

## A.3 Partial Differential Equations

Partial Differential Equations (PDE) are differential equations which include function includes the partial derivatives of two or more variables. Example of such equation is

$$F(u_t, u_x, \ldots) = 0 \tag{A.111}$$

Where subscripts refers to derivative based on it. For example, $u_x = \frac{\partial u}{\partial x}$. Note that partial derivative also include mix of derivatives such as $u_x y$. As one might expect PDE are harder to solve.

Many situations in fluid mechanics can be described by PDE equations. Generally, the PDE solution is done by transforming the PDE to one or more ODE. Partial differential equations are categorized by the order of highest derivative. The nature of the solution is based whether the equation is elliptic parabolic and hyperbolic. Normally, this characterization is done for for second order. However, sometimes similar definition can be applied for other order. The physical meaning of the these definition is that these equations have different characterizations. The solution of elliptic equations depends on the boundary conditions The solution of parabolic equations depends on the boundary conditions but as well on the initial conditions. The hyperbolic equations are associated with method of characteristics because physical situations depends only on the initial conditions. The meaning for initial conditions is that of solution depends on some early points of the flow (the solution). The general second-order PDE in two independent variables has the form

$$a_{xx}\, u_{xx} + 2a_{xy}\, u_{xy} + a_{yy}\, u_{yy} + \cdots = 0 \tag{A.112}$$

The coefficients $a_{xx}$, $a_{xy}$, $a_{yy}$ might depend upon "x" and "y". Equation (A.112) is similar to the equations for a conic geometry:

$$a_{xx}\, x^2 + a_{xy}\, x\, y + a_{yy}\, y^2 + \cdots = 0 \tag{A.113}$$

In the same manner that conic geometry equations are classified are based on the discriminant $a_{xy}^2 - 4\, a_{xx}\, a_{yy}$, the same can be done for a second-order PDE. The discriminant can be function of the $x$ and $y$ and thus can change sign and thus the characteristic of the equation. Generally, when the discriminant is zero the equation are called parabolic. One example of such equation is heat equation. When the discriminant is larger then zero the equation is referred as hyperbolic equations. In fluid mechanics this kind equation appear in supersonic flow or in supper critical flow in open channel flow. The equations that not mentioned above are elliptic which appear in ideal flow and subsonic flow and sub critical open channel flow.

### A.3.1 First-order equations

First order equation can be written as

$$u = a_x \frac{\partial u}{\partial x} + a_y \frac{\partial u}{\partial x} + \ldots \tag{A.114}$$

The interpretation the equation characteristic is complicated. However, the physics dictates this character and will be used in the book.

An example of first order equation is

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{A.115}$$

The solution is assume to be $u = Y(y)\, X(x)$ and substitute into the (A.115) results in

$$Y(y)\, \frac{\partial X(x)}{\partial x} + X(x)\, \frac{\partial Y(y)}{\partial y} = 0 \tag{A.116}$$

Rearranging equation (A.116) yields

$$\frac{1}{X(x)} \frac{\partial X(x)}{\partial x} + \frac{1}{Y(y)} \frac{\partial Y(y)}{\partial y} = 0 \tag{A.117}$$

A possible way the equation (A.117) can exist is that these two term equal to a constant. Is it possible that these terms not equal to a constant? The answer is no if the assumption of the solution is correct. If it turned that assumption is wrong the ratio is not constant. Hence, the constant is denoted as $\lambda$ and with this definition the PDE is reduced into two ODE. The first equation is $X$ function

$$\frac{1}{X(x)} \frac{\partial X(x)}{\partial x} = \lambda \tag{A.118}$$

The second ODE is for $Y$

$$\frac{1}{Y(y)} \frac{\partial Y(y)}{\partial y} = -\lambda \tag{A.119}$$

Equations (A.119) and (A.118) are ODE that can be solved with the methods described before for certain boundary condition.

## A.4 Trigonometry

These trigonometrical identities were set up by Keone Hon with slight modification

1. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
2. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
3. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
4. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
5. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
6. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
2. $\cos 2\alpha = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
3. $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$
4. $\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$ (determine whether it is + or - by finding the quadrant that $\dfrac{\alpha}{2}$ lies in)
5. $\cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ (same as above)
6. $\tan\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1+\cos\alpha}$

for formulas 3-6, consider the triangle with sides of length $a$, $b$, and $c$, and opposite angles $\alpha$, $\beta$, and $\gamma$, respectively

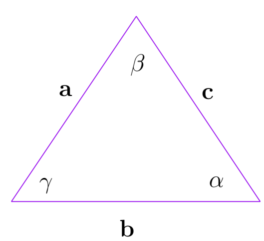


*Fig. -A.7. The tringle angles sides.*

1. $\sin^2\alpha = \dfrac{1-2\cos(2\alpha)}{2}$
2. $\cos^2\alpha = \dfrac{1+2\cos(2\alpha)}{2}$
3. $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$ (Law of Sines)
4. $c^2 = a^2 + b^2 - 2\,a\,b\cos\gamma$ (Law of Cosines)
5. Area of triangle $= \frac{1}{2}a\,b\sin\gamma$
6. Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$ (Heron's Formula)

# Subjects Index

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